$$u = \frac{1 - z\bar{z}}{1 + z\bar{z}}$$

$$v = \frac{z + \bar{z}}{1 + z\bar{z}}$$

$$w = \frac{i\left(\bar{z} - z\right)}{1 + z\bar{z}}$$

$$\frac{1 + z\bar{z}}{2}\frac{\partial}{\partial u} = -z\partial - \bar{z}\,\bar{\partial}$$

$$\frac{1 + z\bar{z}}{2}\frac{\partial}{\partial v} = \partial + \bar{\partial}$$

$$\frac{1 + z\bar{z}}{2i}\frac{\partial}{\partial w} = \partial - \bar{\partial}$$

$$\frac{\partial z}{\partial u} = -\frac{v + iw}{(1 + u)^2} = \frac{-2z}{1 + z\bar{z}}$$

$$\frac{\partial \bar{z}}{\partial u} = -\frac{v - iw}{(1 + u)^2} = \frac{-2\bar{z}}{1 + z\bar{z}}$$

$$\frac{\partial z}{\partial v} = \frac{1}{1 + u} = \frac{2}{1 + z\bar{z}} = \frac{\partial \bar{z}}{\partial v}$$

$$\frac{\partial z}{\partial w} = \frac{i}{1 + u} = \frac{2i}{1 + z\bar{z}} = -\frac{\partial \bar{z}}{\partial w}$$

$$\overbrace{v\frac{\beta+\bar{\beta}}{-2}-w\frac{\beta-\bar{\beta}}{2i}}\frac{\partial}{\partial u}+\overbrace{u\frac{\beta+\bar{\beta}}{2}-2w\alpha i}\frac{\partial}{\partial v}+\overbrace{u\frac{\beta-\bar{\beta}}{2i}+2v\alpha i}\frac{\partial}{\partial w}$$

$$\mathcal{L}=\overbrace{\frac{z+\bar{z}}{1+z\bar{z}}\frac{\beta+\bar{\beta}}{-2}-\frac{i\left(\bar{z}-z\right)}{1+z\bar{z}}\frac{\beta-\bar{\beta}}{2i}}\frac{\partial}{\partial u}+\overbrace{\frac{1-z\bar{z}}{1+z\bar{z}}\frac{\beta+\bar{\beta}}{2}-2\frac{i\left(\bar{z}-z\right)}{1+z\bar{z}}\alpha i}\frac{\partial}{\partial v}+\overbrace{\frac{1-z\bar{z}}{1+z\bar{z}}\frac{\beta-\bar{\beta}}{2i}+2\frac{z+\bar{z}}{1+z\bar{z}}\alpha i}\frac{\partial}{\partial w}$$

$$=\frac{1}{1+z\bar{z}}\left(2i\underbrace{z\bar{\beta}-\bar{z}\beta}\frac{\partial}{\partial u}+\overbrace{\underbrace{1-z\bar{z}}\frac{\beta+\bar{\beta}}{2}-2i\underbrace{\bar{z}-z}\alpha i}\frac{\partial}{\partial v}+\overbrace{\underbrace{1-z\bar{z}}\frac{\beta-\bar{\beta}}{2i}+2\underbrace{z+\bar{z}}\alpha i}\frac{\partial}{\partial w}\right)$$

$$\Rightarrow\overbrace{\frac{1+z\bar{z}}{2}}^{2}\mathcal{L}=\frac{1+z\bar{z}}{2}\left(2i\underbrace{z\bar{\beta}-\bar{z}\beta}\frac{\partial}{\partial u}+\overbrace{\underbrace{1-z\bar{z}}\frac{\beta+\bar{\beta}}{2}-2i\underbrace{\bar{z}-z}\alpha i}\frac{\partial}{\partial v}+\overbrace{\underbrace{1-z\bar{z}}\frac{\beta-\bar{\beta}}{2i}+2\underbrace{z+\bar{z}}\alpha i}\frac{\partial}{\partial w}\right)$$

$$=2i\underbrace{\bar{z}\beta-z\bar{\beta}}\underbrace{z\partial+\bar{z}\bar{\partial}}+\overbrace{\underbrace{1-z\bar{z}}\frac{\beta+\bar{\beta}}{2}-2i\underbrace{\bar{z}-z}\alpha i}\underbrace{\partial+\bar{\partial}}+i\overbrace{\underbrace{1-z\bar{z}}\frac{\beta-\bar{\beta}}{2i}+2\underbrace{z+\bar{z}}\alpha i}\underbrace{\partial-\bar{\partial}}$$

$$=2i\underbrace{\bar{z}\beta-z\bar{\beta}}\underbrace{z\partial+\bar{z}\bar{\partial}}+2i\underbrace{1-z\bar{z}}\underbrace{\beta\partial-\bar{\beta}\bar{\partial}}+4\alpha\underbrace{\bar{z}\bar{\partial}-z\partial}$$

$$=2i\underbrace{\beta-z\bar{\beta}z}\partial-2i\underbrace{\bar{\beta}-\bar{z}\beta\bar{z}}\bar{\partial}+4\alpha\underbrace{\bar{z}\bar{\partial}-z\partial}=\overbrace{\underbrace{\beta+z\bar{\beta}z}-4\alpha z}\partial+\overbrace{4\alpha\bar{z}+\underbrace{\bar{\beta}+\bar{z}\beta\bar{z}}}\bar{\partial}$$