$$\overset{\tau}{\underset{\alpha\beta}{}}\begin{bmatrix}\flat\\ \natural\end{bmatrix} = \sum_{n}^{\mathbb{Z}} {}^{\pi i}\left(\overbrace{n+\alpha/2+\flat}^{2} + \underbrace{2n+\alpha}\underbrace{\natural+\beta/2} + \flat\natural\right)$$

$$\sum_{n}^{\mathbb{Z}} {}^{\pi i}\left(\overbrace{n+\flat}^{2} + 2n\natural + \flat\natural\right) + {}^{\pi i}\left(\overbrace{n+\flat}^{2} + 2n\underbrace{\natural+1/2} + \flat\natural\right)$$

$$+ {}^{\pi i}\left(\overbrace{n+1/2+\flat}^{2} + \underbrace{2n+1}\natural + \flat\natural\right) + {}^{\pi i}\left(\overbrace{n+1/2+\flat}^{2} + \underbrace{2n+1}\underbrace{\natural+1/2} + \flat\natural\right)$$

$$= \sum_{n}^{\mathbb{Z}} {}^{\pi i}\left(\overbrace{n+\flat}^{2} + 2n\natural + \flat\natural\right) + {}^{\pi i}\left(\overbrace{n+\flat}^{2} + 2n\natural + \flat\natural\right)(-1)^{n}$$

$$+ {}^{\pi i}\left(\overbrace{n+1/2+\flat}^{2} + \underbrace{2n+1}\natural + \flat\natural\right) + {}^{\pi i}\left(\overbrace{n+1/2+\flat}^{2} + \underbrace{2n+1}\natural + \flat\natural\right)(-1)^{n+1/2}$$

$$\prod_{I}^{8k}\sum_{n}^{\mathbb{Z}} {}^{\pi i}\left(\overbrace{n+\flat^{I}}^{2} + 2n\natural^{I} + \flat^{I}\natural^{I}\right) + {}^{\pi i}\left(\overbrace{n+\flat^{I}}^{2} + 2n\natural^{I} + \flat^{I}\natural^{I}\right)(-1)^{n}$$

$$+ {}^{\pi i}\left(\overbrace{n+1/2+\flat^{I}}^{2} + \underbrace{2n+1}\natural^{I} + \flat^{I}\natural^{I}\right) + {}^{\pi i}\left(\overbrace{n+1/2+\flat^{I}}^{2} + \underbrace{2n+1}\natural^{I} + \flat^{I}\natural^{I}\right)(-1)^{n+1/2}$$

$$= \sum_{n^{I}}^{\mathbb{Z}^{8k}} {}^{\pi i}\left(\overbrace{n^{I}+\flat^{I}}^{2} + 2n^{I}\natural^{I} + \flat^{I}\natural^{I}\right) + {}^{\pi i}\left(\overbrace{n^{I}+\flat^{I}}^{2} + 2n^{I}\natural^{I} + \flat^{I}\natural^{I}\right)(-1)^{\sum_{I} n^{I}}$$

$$+ {}^{\pi i}\left(\overbrace{n^{I}+1/2+\flat^{I}}^{2} + 2\left(n^{I}+1/2\right)\natural^{I} + \flat^{I}\natural^{I}\right) + {}^{\pi i}\left(\overbrace{n^{I}+1/2+\flat^{I}}^{2} + 2\left(n^{I}+1/2\right)\natural^{I} + \flat^{I}\natural^{I}\right)(-1)^{\sum_{I}(n^{I}+1/2)}$$

$$= \sum_{n^{I}}^{\mathbb{Z}_{+}^{8k}} {}^{\pi i}\left(\overbrace{n^{I}+\flat^{I}}^{2} + 2n^{I}\natural^{I} + \flat^{I}\natural^{I}\right) + {}^{\pi i}\left(\overbrace{n^{I}+1/2+\flat^{I}}^{2} + 2\underbrace{n^{I}+1/2}\natural^{I} + \flat^{I}\natural^{I}\right)$$

$$= \sum_{m^{I}}^{\mathbb{Z}_{+}^{8k} \cup (\mathbb{Z}+1/2)_{+}^{8k}} {}^{\pi i}\left(\overbrace{m^{I}+\flat^{I}}^{2} + 2m^{I}\natural^{I} + \flat^{I}\natural^{I}\right)$$

ev self-dual lattice $\mathfrak{t}_{\mathbb{Z}} = \mathbb{Z}_{+}^{8k} \cup \overbrace{\mathbb{Z}+1/2}_{+}^{8k} = \dfrac{m^{I} \in \mathbb{Z}^{8k} \cup \overbrace{\mathbb{Z}+1/2}^{8k}}{\sum_{I} m^{I} \in 2\mathbb{Z}} \sqsubset \mathbb{R}^{8k}$

$$\sum_{m^I}^{\mathbb{Z}_+^{8k} \cup \overbrace{\mathbb{Z}+1/2}_+^{8k}} {}^{\pi i} \left(\overbrace{m^I + \flat^{\mu}{}_{\mu} \cancel{1}^{I}}^{2} - 2m^I \llcorner^{\mu}{}_{\mu} \cancel{1}^{I} - \flat^{\mu}{}_{\mu} \cancel{1}^{I} \llcorner^{\mu}{}_{\mu} \cancel{1}^{I} \right)$$