

$$\mathbb{R} \text{ basis } {}^n\mathbb{C} \xleftarrow[\text{inj}]{\Gamma} {}^{2n}\mathbb{R} \Leftrightarrow \begin{bmatrix} \Gamma \\ \bar{\Gamma} \end{bmatrix} \in {}^{2n}\mathbb{C}_{2n}^{\mathbb{C}} \text{ inv}$$

$$\Rightarrow : \quad \mathbb{J} \in {}^{2n}\mathbb{C} : \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Gamma \\ \bar{\Gamma} \end{bmatrix} \mathbb{J} = \begin{bmatrix} \Gamma \mathbb{J} \\ \bar{\Gamma} \mathbb{J} \end{bmatrix} \Rightarrow \Gamma \mathbb{J} = 0 = \bar{\Gamma} \mathbb{J}$$

$$\Rightarrow \Gamma \underbrace{\mathbb{J} + \bar{\mathbb{J}}}_{=} = \Gamma \mathbb{J} + \Gamma \bar{\mathbb{J}} = \Gamma \mathbb{J} + \overline{\bar{\Gamma} \mathbb{J}} = 0 \xRightarrow{\text{Vor}} {}^{2n}\mathbb{R} \ni \mathbb{J} + \bar{\mathbb{J}} = 0$$

$$\Gamma \underbrace{\mathbb{J} - \bar{\mathbb{J}}}_{=} = \Gamma \mathbb{J} - \Gamma \bar{\mathbb{J}} = \Gamma \mathbb{J} - \overline{\bar{\Gamma} \mathbb{J}} = 0 \Rightarrow \Gamma i \underbrace{\mathbb{J} - \bar{\mathbb{J}}}_{=} = i \Gamma \underbrace{\mathbb{J} - \bar{\mathbb{J}}}_{=} = 0 \xRightarrow{\text{Vor}} {}^{2n}\mathbb{R} \ni i \underbrace{\mathbb{J} - \bar{\mathbb{J}}}_{=} = 0$$

$$\Rightarrow \mathbb{J} = \frac{\mathbb{J} + \bar{\mathbb{J}}}{2} + \frac{i \underbrace{\mathbb{J} - \bar{\mathbb{J}}}_{=}}{2i} = 0$$

$$\Leftarrow : \quad \mathbb{J} \in {}^{2n}\mathbb{R} : \quad \Gamma \mathbb{J} = 0$$

$$\Rightarrow \bar{\Gamma} \mathbb{J} = \bar{\Gamma} \bar{\mathbb{J}} = \overline{\Gamma \mathbb{J}} = 0 \Rightarrow \begin{bmatrix} \Gamma \\ \bar{\Gamma} \end{bmatrix} \mathbb{J} = \begin{bmatrix} \Gamma \mathbb{J} \\ \bar{\Gamma} \mathbb{J} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbb{J} = 0$$