q-binomial

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \prod_i^m \frac{q^n - q^i}{q^m - q^i} = \prod_i^m \frac{q^{n-i} - 1}{q^{m-i} - 1}$$

$$= \frac{q^n - 1}{q^m - 1} \frac{q^n - q}{q^m - q} \cdot\cdot \frac{q^n - q^{m-}}{q^m - q^{m-}} = \frac{q^n - 1}{q^m - 1} \frac{q^{n-1} - 1}{q^{m-1} - 1} \cdot\cdot \frac{q^{n-m+1} - 1}{q - 1}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m-1 \end{bmatrix}_q \frac{q^{n-m+1} - 1}{q^m - 1}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{q^n - 1}{q^m - 1} \frac{q^{n-1} - 1}{q^{m-1} - 1} \cdot\cdot \frac{q^{n-m+1} - 1}{q - 1}$$

$$= \frac{q^n - 1}{q^{m-1} - 1} \frac{q^{n-1} - 1}{q^{m-2} - 1} \cdot\cdot \frac{q^{n-m+2} - 1}{q - 1} \frac{q^{n-m+1} - 1}{q - 1} = \begin{bmatrix} n \\ m-1 \end{bmatrix}_q \frac{q^{n-m+1} - 1}{q^m - 1}$$

$$m \in 1|n: \quad \begin{bmatrix} n+1 \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m \end{bmatrix}_q + q^{n+1-m} \begin{bmatrix} n \\ m-1 \end{bmatrix}_q$$

$$\prod_{i}^{n}\left(x+q^{i}\right)=\sum_{m}^{0|n}\begin{bmatrix}n\\m\end{bmatrix}_q q^{m\,(m-1)/2}\,x^{n-m}$$

$$\prod_{i}^{0|n}\left(x+q^{i}\right)=(x+q^{n})\prod_{i}^{n}\left(x+q^{i}\right)=\sum_{m}^{0|n}\begin{bmatrix}n\\m\end{bmatrix}_q q^{m\,(m-1)/2}\,x^{n-m}\,(x+q^{n})$$

$$=\sum_{m}^{0|n}\begin{bmatrix}n\\m\end{bmatrix}_q q^{m\,(m-1)/2}\,x^{n-m+1}+\sum_{k}^{0|n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{n+k\,(k-1)/2}\,x^{n-k}$$

$$=\sum_{m}^{0|n}\begin{bmatrix}n\\m\end{bmatrix}_q q^{m\,(m-1)/2}\,x^{n-m+1}+\sum_{m}^{1|n+1}\begin{bmatrix}n\\m-1\end{bmatrix}_q q^{n+(m-1)\,(m-2)/2}\,x^{n+1-m}$$

$$=\sum_{m}^{1|n}\left(\begin{bmatrix}n\\m\end{bmatrix}_q+q^{n-m+1}\begin{bmatrix}n\\m-1\end{bmatrix}_q\right)q^{m\,(m-1)/2}\,x^{n+1-m}+\begin{bmatrix}n\\0\end{bmatrix}_q q^{0}\,x^{n+1}+\begin{bmatrix}n\\n\end{bmatrix}_q q^{n+n\,(n-1)/2}\,x^{0}$$

$$=\sum_{m}^{1|n}\begin{bmatrix}n+1\\m\end{bmatrix}_q q^{m\,(m-1)/2}\,x^{n+1-m}+\begin{bmatrix}n+1\\0\end{bmatrix}_q q^{0}\,x^{n+1}+\begin{bmatrix}n+1\\n+1\end{bmatrix}_q q^{(n+1)\,n/2}\,x^{0}$$

$$=\sum_{m}^{0|n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_q q^{m\,(m-1)/2}\,x^{n+1-m}$$

$$\frac{q^{n}-1}{q^{m}-1}\rightsquigarrow\frac{n}{m}$$

$$\frac{q^{n}-1}{q^{m}-1}=\frac{q^{n}-1}{q-1}\,\frac{q-1}{q^{m}-1}=\frac{q^{n-1}+q^{n-2}+\cdot\cdot+q^{0}}{q^{m-1}+q^{m-2}+\cdot\cdot+q^{0}}\rightsquigarrow\frac{n}{m}$$

$$\text{Hosp}:\quad\frac{q^{n}-1}{q^{m}-1}\equiv\frac{nq^{n-1}}{mq^{m-1}}=\frac{n}{m}\,q^{n-m}\rightsquigarrow\frac{n}{m}$$

$$\frac{q^{n}-q^{i}}{q^{m}-q^{i}}\rightsquigarrow\frac{n-i}{m-i}$$

$$\text{LHS}\;=\frac{q^{n-i}-1}{q^{m-i}-1}\rightsquigarrow\;\text{RHS}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q \rightsquigarrow \begin{bmatrix} n \\ m \end{bmatrix}$$

$$\text{LHS} \;= \frac{q^n - 1}{q^m - 1}\frac{q^n - q}{q^m - q}\cdot\cdot\frac{q^n - q^{m-}}{q^m - q^{m-}} \rightsquigarrow \frac{n}{m}\frac{n-1}{m-1}\frac{n-m+1}{1} = \;\text{RHS}$$

$$\prod_i^n \left(1 + q^i x\right) = \sum_m^{0|n} \sigma_m \left(q^0 x \cdots q^{n-1} x\right) = \sum_m^{0|n} x^m \, \sigma_m \left(q^0 \cdots q^{n-1}\right)$$

$$\sigma_m \left(q^{0|n-1}\right) = \sum_{0 \leqslant i_1 < \cdots < i_m < n} q^{i_1 + + i_m}$$

$$\sigma_m \left(q^{0|n}\right) = \sum_{0 \leqslant i_1 < \cdots < i_m \leqslant n} q^{i_1} \cdots q^{i_m} = \sum_{0 \leqslant i_1 < \cdots < i_m < n} q^{i_1 + + i_m} + q^n \sum_{0 \leqslant i_1 < \cdots < i_{m-} < n} q^{i_1 + + i_{m-}}$$

$$= \sigma_m \left(q^{0|n-1}\right) + q^n \, \sigma_{m-} \left(q^{0|n-1}\right) = q^{m\,(m-1)\,/2} \begin{bmatrix} n \\ m \end{bmatrix}_q + q^n \, q^{(m-1)\,(m-2)\,/2} \begin{bmatrix} n \\ m-1 \end{bmatrix}_q$$

$$= q^{m\,(m-1)\,/2} \left(\begin{bmatrix} n \\ m \end{bmatrix}_q + q^{n+1-m} \begin{bmatrix} n \\ m-1 \end{bmatrix}_q \right)$$

$$q^{m\,(m-1)\,/2} \begin{bmatrix} n \\ m \end{bmatrix}_q = \prod_i^m q^i \prod_i^m \frac{q^n - q^i}{q^m - q^i} = \prod_i^m q^i \prod_i^m \frac{q^n - q^i}{q^{m-i} - 1}$$

$$\prod_{i}^{m} \underbrace{x - q^i} = \sum_{k}^{0|m} \overset{k}{(-q)} \begin{bmatrix} m \\ k \end{bmatrix} x^{m-k}$$

$$0 \leqslant k \leqslant m \Rightarrow \sigma_k \left(q^0 \cdots q^{m\text{-}1}\right) = \sum_{0 \leqslant i_1 < \cdots < i_k < m} q^{i_1} \cdots q^{i_k} = \sum_{S \subset m}^{\sharp S = k} \prod_{i}^{S} q^i = \sum_{S \subset m}^{\sharp S = k} q^{\sharp S} = q^k \sum_{S \subset m}^{\sharp S = k} 1 = q^k \begin{bmatrix} m \\ k \end{bmatrix}$$

$$\Rightarrow \prod_{i}^{m} \underbrace{x - q^i} = x^m \prod_{i}^{m} \underbrace{1 - \frac{q^i}{x}} = x^m \sum_{k}^{0|m} (-1)^k \sigma_k \left(\frac{q^0}{x} \cdots \frac{q^{m\text{-}1}}{x}\right) = \sum_{k}^{0|m} (-1)^k x^{m\text{-}k} \sigma_k \left(q^0 \cdots q^{m\text{-}1}\right)$$

$$= \sum_{k}^{0|m} (-1)^k x^{m\text{-}k} q^k \begin{bmatrix} m \\ k \end{bmatrix} = \text{RHS}$$