

$$\mathbb{K}_{n+1}\nabla\!\!\!\nabla \xleftarrow{i^{\bar{\Theta}}} \mathbb{K}_n\nabla\!\!\!\nabla$$

$$\begin{cases} {}_i\bar{\Theta}_J\Theta = {}_{iJ}\Theta & i \leqslant J \\ {}_i\bar{\Theta}_J\Theta - {}_{iJ}\Theta \in \mathbb{K}_n\nabla\!\!\!\nabla \\ {}_i\bar{\Theta}_j\bar{\Theta}_K\Theta - {}_j\bar{\Theta}_i\bar{\Theta}_K\Theta = \overline{{}_i\Theta \times {}_j\Theta}_K\Theta & |K| < n \end{cases}$$

$$\mathbb{K}_{n+2}\nabla\!\!\!\nabla \xleftarrow{i^{\bar{\Theta}}} \mathbb{K}_{n+1}\nabla\!\!\!\nabla$$

$${}_i\bar{\bar{\Theta}}_{jK}\Theta = \begin{cases} {}^{ijK}\Theta & i \leqslant j \leqslant K \\ {}_{ijK}\Theta + \overline{{}_i\Theta \times {}_j\Theta}_K\Theta + {}_j\bar{\Theta}_{\underbrace{i\Theta_K\Theta - {}_{iK}\Theta}} & i > j \leqslant K \end{cases}$$

$$L < n \implies {}_i \bar{\oplus} {}_j \bar{\ominus} {}_{kL} \oplus - {}_j \bar{\oplus} {}_i \bar{\ominus} {}_{kL} \oplus = \overline{{}_i \oplus \times {}_j \oplus} {}_{kL} \oplus$$

$$\begin{aligned} j > k \Rightarrow \bar{\theta}_{j-kL} &= \bar{\theta}_j \bar{\theta}_{kL} = \bar{\theta}_k \bar{\theta}_j \bar{\theta}_{L} + \overline{\bar{\theta}_j * \bar{\theta}_k} \bar{\theta}_L = \bar{\theta}_k \bar{\theta}_{jL} + \underbrace{\bar{\theta}_k \bar{\theta}_j \bar{\theta}_L - \bar{\theta}_{jL}}_{= \bar{\theta}_{kjL}} + \overline{\bar{\theta}_j * \bar{\theta}_k} \bar{\theta}_L \\ &= \bar{\theta}_{kjL} + \underbrace{\bar{\theta}_k \bar{\theta}_j \bar{\theta}_L - \bar{\theta}_{jL}}_{= \bar{\theta}_{kjL}} + \overline{\bar{\theta}_j * \bar{\theta}_k} \bar{\theta}_L \end{aligned}$$

$$i > k < j \Rightarrow {}_i \bar{\bar{\mathfrak{g}}}{}_{kjL} \mathfrak{g} = {}_{ikjL} \mathfrak{g} + \overline{{}_i \mathfrak{g} \times {}_k \mathfrak{g}} {}_{jL} \mathfrak{g} + \bar{k} \bar{i} \bar{j} \bar{l} \underbrace{{}_{jl} \mathfrak{g} - {}_{il} \mathfrak{g}}$$

$$\bar{\bar{\bar{g}}}_{i\bar{j}} \bar{\bar{\bar{g}}}_{j\bar{k}} L \bar{\bar{\bar{g}}} = \bar{\bar{\bar{g}}}_{i\bar{k}} \bar{\bar{\bar{g}}}_{kjL} \bar{\bar{\bar{g}}} + \bar{\bar{\bar{g}}}_{i\bar{k}} \bar{\bar{\bar{g}}}_{\underbrace{j\bar{L}}_{\bar{j}} \bar{\bar{\bar{g}}}} - \bar{\bar{\bar{g}}}_{j\bar{L}} \bar{\bar{\bar{g}}} + \bar{\bar{\bar{g}}}_{i\bar{j}} \bar{\bar{\bar{g}}}_{\bar{j}\bar{k}} \times \bar{\bar{\bar{g}}}_{\bar{k}L} L \bar{\bar{\bar{g}}} = \bar{i} \bar{k} j \bar{j} L \bar{\bar{\bar{g}}} + \bar{i} \bar{k} \times \bar{k} \bar{j} L \bar{\bar{\bar{g}}} + \bar{k} \bar{i} \bar{j} \bar{j} L \bar{\bar{\bar{g}}} - \bar{i} \bar{j} \bar{j} L \bar{\bar{\bar{g}}}$$

$$= \underbrace{i_{kjL}\theta_i}_{+} + \underbrace{\theta_i \times \theta_k}_{+} \bar{\theta}_j L\theta_i + \underbrace{\bar{\theta}_k \bar{\theta}_i \bar{\theta}_j \bar{\theta}_L \theta_i}_{-} - i_{jL}\theta_j + \underbrace{\theta_j \times \theta_k}_{+} \bar{\theta}_i L\theta_i + \underbrace{\theta_i \theta_j \theta_k \theta_L}_{+} \theta_i$$

$$\text{LHS} = \underbrace{\bar{\theta}_k \bar{\theta}_i \bar{\theta}_j - \bar{\theta}_j \bar{\theta}_i \bar{\theta}_k}_{L \bar{\theta}} + \overline{\bar{\theta}_i \times \underbrace{\bar{\theta}_j \bar{\theta}_k \bar{\theta}_i}_{L \bar{\theta}}} - \overline{\bar{\theta}_j \bar{\theta}_k \bar{\theta}_i \times \underbrace{\bar{\theta}_i \bar{\theta}_k \bar{\theta}_j}_{L \bar{\theta}}}$$

$$= \bar{\mathbf{A}}_k \overline{\mathbf{B}_i \times \mathbf{B}_j} L \mathbf{B}_k + \overline{\mathbf{B}_i \times \mathbf{B}_j} \times \bar{\mathbf{A}}_k L \mathbf{B}_k = \overline{\mathbf{B}_i \times \mathbf{B}_j} \bar{\mathbf{A}}_k L \mathbf{B}_k = \text{RHS}$$

$$i \leq k < j \Rightarrow \bar{\bar{\Theta}}_i \bar{\bar{\Theta}}_j \bar{\bar{\Theta}}_{kL} = \bar{\bar{\Theta}}_i \bar{\bar{\Theta}}_{kjL} \bar{\bar{\Theta}} + \bar{\bar{\Theta}}_i \bar{\bar{\Theta}}_k \bar{\bar{\Theta}}_{\underbrace{jL \bar{\bar{\Theta}} - jL \bar{\bar{\Theta}}}_{(j-L) \bar{\bar{\Theta}}} + \bar{\bar{\Theta}}_i \bar{\bar{\Theta}}_j \bar{\bar{\Theta}}_k \bar{\bar{\Theta}}_{L} \bar{\bar{\Theta}}$$

$$= \underset{i k j L}{\mathbf{\Theta}} + \underset{i}{\bar{\mathbf{\Theta}}} \underset{k}{\bar{\mathbf{\Theta}}} \underset{j L}{\cancel{\bar{\mathbf{\Theta}} \mathbf{\Theta}}} - \underset{j L}{\cancel{\bar{\mathbf{\Theta}}}} + \underset{i}{\bar{\mathbf{\Theta}}} \underset{j}{\cancel{\mathbf{\Theta} \times}} \underset{k}{\bar{\mathbf{\Theta}}} \underset{L}{\mathbf{\Theta}}$$

$$j \bar{\Theta}^i_k \bar{\Theta}_{kL} = j \bar{\Theta}_{ikL} \Theta + \overbrace{j \Theta^i \times \Theta^i} kL + i \bar{\Theta}^j_k \bar{\Theta}_{kL} - j \bar{\Theta}^k_L$$

$$\text{LHS} = \overbrace{\underset{i}{\cancel{\alpha}} \times \underset{j}{\cancel{\alpha}}}_{kL} \underset{k}{\alpha} + \underset{i}{\cancel{\alpha}} \underbrace{\underset{k}{\cancel{\alpha}} \underset{j}{\cancel{\alpha}} \underset{L}{\alpha} - \underset{jL}{\cancel{\alpha}}}_{+ \overbrace{\underset{j}{\cancel{\alpha}} \times \underset{k}{\cancel{\alpha}} \underset{L}{\alpha} - \underset{j}{\cancel{\alpha}} \underset{kL}{\alpha} + \underset{jkL}{\cancel{\alpha}}}^{= 0}}$$

$$k \underbrace{\theta_j \theta_L \theta_{jL} - \theta_{jL} \theta_j}_{+ j \theta_k \theta_L \theta_{jL} - j \theta_j \theta_{kL} \theta_j + j k L \theta_j} = j \theta_k \theta_L \theta_{jL} - j \theta_j \theta_{kL} \theta_j - k \theta_j \theta_{jL} \theta_j + j k L \theta_j = 0$$