

$${}^r\mathbb{K}_r^{\mathbb{C}}$$

$$\mathbf{C}|\mathbf{L}$$

$${}^r\mathbb{K}_r^U \text{ unitary group}$$

$$U|L$$

$$\det \mathcal{L} = \sum_{\pi \in S_n} {}^{-\pi} 1 {}_1 \mathcal{L}^{\pi_1} \cdots {}_n \mathcal{L}^{\pi_n}$$

$${}^I \underline{\det} \mathcal{L} = \text{tr } \mathcal{L}$$

$$\mathcal{L}_t \in {}_n^{\mathbb{C}}\mathbb{K}^n$$

$$\mathcal{L}_0 = I$$

$$\partial_t^0 \mathcal{L}_t = \mathcal{L}$$

$$\Rightarrow {}^I \underline{\det} \mathcal{L} = \partial_t^0 \det \mathcal{L}_t = \partial_t^0 \sum_{\pi \in S_n} {}^{-\pi} 1 {}_1 \mathcal{L}_t^{\pi_1} \cdots {}_n \mathcal{L}_t^{\pi_n} = \sum_{\pi \in S_n} {}^{-\pi} 1 \partial_t^0 {}_1 \mathcal{L}_t^{\pi_1} \cdots {}_n \mathcal{L}_t^{\pi_n}$$

$$= \sum_{\pi \in S_n} {}^{-\pi} 1 \underbrace{\partial_t^0 {}_1 \mathcal{L}_t^{\pi_1} \cdots {}_n \mathcal{L}_t^{\pi_n} + \cdots + {}_1 \mathcal{L}_t^{\pi_1} \cdots \partial_t^0 {}_n \mathcal{L}_t^{\pi_n}}$$

$$= \sum_{\pi \in S_n} {}^{-\pi} 1 \underbrace{{}_1 \mathcal{L}^{\pi_1} \cdots {}_n \delta^{\pi_n} + \cdots + {}_1 \delta^{\pi_1} \cdots {}_n \mathcal{L}^{\pi_n}} = {}_1 \mathcal{L}^1 + \cdots + {}_n \mathcal{L}^n = \text{tr } \mathcal{L}$$

$$\frac{\mathfrak{b}_{\overbrace{\mathcal{L}}^{\cdot}}}{\mathfrak{b}_{\underbrace{\mathcal{L}}_{\cdot}}} = \overbrace{\mathfrak{b}_{\mathcal{L}} \bar{\mathcal{L}}^1}^{\cdot} : \quad \mathfrak{b}_{\overbrace{\mathcal{L}}^{\cdot}} = \overbrace{\mathcal{L}}^{\cdot} \overbrace{\mathfrak{b}_{\mathcal{L}} \bar{\mathcal{L}}^1}^{\cdot}$$

$$\begin{aligned} \mathfrak{b}_{\overbrace{\mathcal{L}}^{\cdot}} &= \mathfrak{b} \sum_{\pi} -1 \cdot {}_1 \mathcal{L}^{\pi_1} \cdots {}_n \mathcal{L}^{\pi_n} = \sum_{\pi} -1 \overbrace{{}_1 \mathcal{L}^{\pi_1} \cdots {}_n \mathcal{L}^{\pi_n}}^{\cdot} + \cdots + \overbrace{{}_1 \mathcal{L}^{\pi_1} \cdots {}_n \mathcal{L}^{\pi_n}}^{\cdot} \\ &= \sum_{\pi} -1 \sum_k {}_1 \mathcal{L}^{\pi_1} \cdots \underbrace{{}_k \mathcal{L}^{\pi_k}}_{\cdot} \cdots {}_n \mathcal{L}^{\pi_n} \\ &= \sum_k \sum_{\ell} \underbrace{\mathfrak{b}_k \mathcal{L}^{\ell}}_{\cdot} \sum_{\pi_k = \ell} -1 \cdot {}_1 \mathcal{L}^{\pi_1} \wedge \cdots \wedge {}_n \mathcal{L}^{\pi_n} = \sum_k \sum_{\ell} \underbrace{\mathfrak{b}_k \mathcal{L}^{\ell}}_{\cdot} -1^{k+\ell} \sum_{\sigma} -1 \cdot {}_1 \mathcal{L}^{\sigma_1} \wedge \cdots \wedge {}_n \mathcal{L}^{\sigma_n} \\ &= \sum_k \sum_{\ell} \underbrace{\mathfrak{b}_k \mathcal{L}^{\ell}}_{\cdot} -1^{k+\ell} \underbrace{{}_{N \sqcup k} \mathcal{L}^{N \sqcup \ell}}_{\cdot} = \underbrace{\mathcal{L}}_{\cdot} \sum_k \sum_{\ell} \underbrace{\mathfrak{b}_k \mathcal{L}^{\ell}}_{\cdot} \bar{\mathcal{L}}^{1_k} = \underbrace{\mathcal{L}}_{\cdot} \sum_k \underbrace{\mathfrak{b}_k \mathcal{L}}_{\cdot} \bar{\mathcal{L}}^{1_k} = \underbrace{\mathcal{L}}_{\cdot} \overbrace{\mathfrak{b}_{\mathcal{L}} \bar{\mathcal{L}}^1}^{\cdot} \end{aligned}$$

$$N \sqcup k \xrightarrow{\text{bij}} \sigma N \sqcup \ell$$

$$\begin{aligned} -1^{\pi} &= -1^{\sigma} -1^{k+\ell} \\ -1^{\pi} &= \prod_{i < j} \pi_i \# \pi_j = \prod_{k \neq i < j \neq k} \pi_i \# \pi_j \prod_{i < k} \pi_i \# \ell \prod_{k < j} \ell \# \pi_j \\ \prod_{k \neq i < j \neq k} \pi_i \# \pi_j &= -1^{\sigma} \\ \prod_{i < k} \pi_i \# \ell \prod_{k < j} \ell \# \pi_j &= -1^{k-1} \prod_{i < k} \ell \# \pi_i \prod_{k < j} \ell \# \pi_j = -1^{k-1} \prod_{m \neq k} \ell \# \pi_m = -1^{k-1} -1^{\ell-1} = -1^{k+\ell} \end{aligned}$$