Quantization and Analysis on Symmetric Spaces

Proceedings of the ESI workshops

Quantization, Complex and Harmonic Analysis
Complex Analysis and Operator Theory on Symmetric Spaces

Harald Upmeier (editor)

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Preface

Harald Upmeier

As part of the special program "Complex Analysis and Applications", Fall 2005, at the Erwin Schrödinger Institute, organized by F. Haslinger, E. Straube and H. Upmeier, two mini-workshops

Quantization, Complex and Harmonic Analysis

and

Complex Analysis and Operator Theory on Symmetric Spaces

were held during September 22–23, 2005 and on October 12, 2005, resp. These workshops were devoted to interactions between complex analysis on hermitian symmetric spaces and other areas of mathematics and mathematical physics, notably harmonic analysis on semisimple Lie groups and quantization theory on Kähler manifols.

This research area has been developing considerably during the past few years, and has now reached a certain maturity from where new research directions, such as infinite dimensional Hilbert symmetric manifolds or analysis on vector-valued holomorphic functions and more general discrete series representations can be actively pursued.

The talks given at the two workshops provide an overall picture of the current status of the field, and it was agreed to collect short expository articles from the participants as (informal) Proceedings. In addition, several participants have published separate research papers in the ESI preprint series.

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Index Theory for Wiener-Hopf Operators on Convex Cones

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We report on work in progress, conducted jointly with Troels R. Johansen (Paderborn University, Germany). We study Wiener-Hopf operators associated to finite-dimensional convex cones in Euclidean space. We determine a composition series for the C*-algebra generated by Wiener-Hopf operators with integrable symbols, and embark on a detailed study of the index maps induced by its subquotients.

1 Motivation

1.1 The classical Wiener-Hopf equation

1.1.1. The classical Wiener-Hopf equation is of the form $(1+W_f)u = v$, where

$$W_f u(x) = \int_0^\infty f(x-y)u(y)\,dy \quad \text{for all} \quad f \in \mathbf{L}^1(\mathbb{R})\,,\ u \in \mathbf{L}^2(0,\infty)\,,\ x \in [0,\infty[\ .$$

The bounded operator W_f is called the Wiener-Hopf operator of symbol f. The operator W_f is conjugate, via the Euclidean Fourier transform, to the Toeplitz operator $T_{\hat{f}}$ defined on the Hardy space of the upper half plane, and thus has connection to both complex and harmonic analysis. The one-variable WH equation is well understood, by the following classical theorem [GK58].

Theorem 1.1.2. Let $\mathcal{W}(0,\infty)$ be the C*-algebra generated by W_f, $f \in \mathbf{L}^1(\mathbb{R})$.

(i). The following sequence is exact

$$0 \longrightarrow \mathbb{K}(\mathbf{L}^2(0,\infty)) \longrightarrow \mathcal{W}(0,\infty) \xrightarrow{\sigma} \mathcal{C}_0(\mathbb{R}) \longrightarrow 0$$

where σ is the Wiener-Hopf representation, defined by $\sigma(W_f) = \hat{f}$.

(ii). $1+W_f$ is Fredholm if and only if $1+\hat{f}$ is everywhere $\neq 0$ on $\mathbb{R}^+ = \mathbb{S}^1$.

(iii). In this case, $\operatorname{Index}(1+W_f)$ is the negative winding number of $1+\hat{f}$ around 0.

1.2 Multivariate generalisation

1.2.1. It is quite straightforward to generalise the above setting to several variables. Indeed, let X be a finite-dimensional real vector space endowed with some Euclidean inner product $(\sqcup : \sqcup)$, and let $\Omega \subset X$ be a closed, pointed and solid convex cone. I.e., Ω contains no line, and has non-void interior. Consider Lebesgue measure on X to define $\mathbf{L}^1(X)$ and its restriction to Ω to define $\mathbf{L}^2(\Omega)$.

Then, W_f is defined by

$$W_f u(x) = \int_\Omega f(x-y)u(y)\,dy \quad \text{for all} \quad f \in \mathbf{L}^1(X)\,,\; u \in \mathbf{L}^2(\Omega)\,,\; x \in \Omega\;.$$

Moreover, let $\mathcal{W}(\Omega)$ be the *Wiener-Hopf algebra*, the C*-subalgebra of all bounded operators on $\mathbf{L}^2(\Omega)$ generated by the collection of the W_f, $f \in \mathbf{L}^1(\Omega)$.

The programme we propose to study then is the following:

(1). Determine a composition series of $\mathcal{W}(\Omega)$ and compute its subquotients.

(2). Find Fredholmness criteria for Wiener-Hopf operators.

(3). Give an index formula which expresses their Fredholm (family) index in terms of topological data.

These problems have been addressed from different angles in a quite extensive literature. Pioneering work was done in the series of papers by Coburn-Douglas [CD69, CD71], partly jointly with Schaeffer and Singer [CDSS71, CDS72]. Together with the work of Douglas-Howe [DH71], this culminated in the solution of problems (1) and (3) for the example of the (discrete) quarter plane. Berger-Coburn [BC79] were the first to address the structure of the Hardy-Toeplitz algebra (equivalent to the WH algebra for symmetric tube type domains) for a symmetric domains of rank 2, the 2×2 matrix ball (the rank 1 case having been essentially solved by Venugopalkrishna). This led to the paper of Berger-Coburn-Korányi [BCK80] which treats the case of all Lorentz cones (also corresponding to rank 2 symmetric domains, the Lie balls).

Major advances were made by Upmeier [Upm84, Upm88b, Upm88a] who solved the problem (1) for the Hardy-Toeplitz algebras of all bounded symmetric domains (which properly include the WH algebras for symmetric cones). Moreover, he developed an index theory, proving index formulae for the all Wiener-Hopf operators associated to symmetric cones, thus solving problem (3) for this class of cones. A basic tool in his approach is the Cayley transform, which allows for the transferral to the situation of bounded symmetric domains.

Another approach was taken by Dynin [Dyn86], who uses an inductive procedure, based on the local decomposition of the cone Ω into a product relative to a fixed exposed face, for the construction of the composition series as in (1). This presumes a certain tameness of the cone Ω, which he calls 'complete tangibility'. Due to the weakness of this assumption, a large class of cones, including polyhedral, almost smooth and homogeneous cones, are subsumed.

The point of view we will adopt in this note is due to Muhly and Renault [MR82]. They describe a general procedure to produce a (locally compact, measured) groupoid whose groupoid C*-algebra (a generalised group C*-algebra) is just the WH algebra, and compute composition series (1) for the opposite extremes of polyhedral and symmetric cones. Their construction is based on the specification of a convenient compactification of Ω (in fact, of X). Nica gives a uniform construction of this WH compactification for *all* pointed and solid cones. The main problem is to prove that the corresponding groupoid always has a Haar system. From the more general perspective of causal homogeneous spaces, in which X is replaced by a locally compact group and Ω by a submonoid satisfying certain assumptions, Hilgert-Neeb extended Nica's results, at the same time giving a convenient alternative description of the WH compactification.

As yet, none of the problems (1)-(3) have been solved in full generality. In fact, there is not even an index theorem for the simplicial case. We show how the groupoid perspective allows for a unified treatment of problems (1) and (3), for a very large class of cones satisfying some global regularity assumption which arises in a natural fashion.

2 Groupoid approach to Wiener-Hopf operators

2.1 Why groupoids?

2.1.1. Viewed at arms length, Wiener-Hopf operators are just a some kind restricted convolution operators. On the other hand, groupoids are the rigorous formulation of the vague concept of a group with a partially defined group law. Hence, we may suspect that they provide a domain for Wiener-Hopf operators to grow and thrive. Let us make this more precise.

2.1.2. A (locally compact, measured) groupoid $\mathcal{G}$ is the prescription of the following data:

(G1). The sets $\mathcal{G}$ of arrows and $\mathcal{G}^{(0)}$ of units, the latter injected into the former by the map $x \mapsto \mathrm{id}_x$; two left inverses $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ to this map, defining *range* and *source* of all arrows; a composition $\circ : \mathcal{G}^{(2)} \to G$, defined on the set of composable pairs (γ_1, γ_2) where $r(\gamma_1) = s(\gamma_2)$, such that $r(\gamma_2 \circ \gamma_1) = r(\gamma_2)$, $s(\gamma_2 \circ \gamma_1) = s(\gamma_1)$, the units id_x are units for $\circ$, $\circ$ is associative, and any arrow is invertible.

Put more succinctly, $\mathcal{G}$ is a small category in which all arrows are isomorphisms.

(G2). A locally compact topology for which $\circ$ and $\sqcup^{-1}$ are continuous.

(G3). A *Haar system*, which is a weakly continuous map $\lambda : x \mapsto \lambda^x$ from $\mathcal{G}^{(0)}$ to the set of positive Radon measures on $\mathcal{G}$, such that $\operatorname{supp} \lambda^x = r^{-1}(x)$ for all $x \in \mathcal{G}^{(0)}$ and $\gamma(\lambda^{s(\gamma)}) = \lambda^{r(\gamma)}$ for all $\gamma \in \mathcal{G}$.

One should note that the existence of a Haar system does not follow from conditions (G1) and (G2). Indeed, condition (G3) implies that the maps r and s are open onto $\mathcal{G}^{(0)}$. Moreover, even if a Haar system exists, it may be far from unique. In fact, if Y is any locally compact space, then $Y \times Y$ is in a unique fashion a groupoid, such that the projections onto the two factors are source and range. Then, any fully supported Radon measure α on Y defines a Haar system by the prescription $\lambda^x = \delta_x \otimes \alpha$.

Given a groupoid $\mathcal{G}$ with Haar system λ, we may define convolution on the set $\mathcal{K}(\mathcal{G})$ of compactly supported continuous functions on $\mathcal{G}$, as follows

$$\varphi * \psi(\gamma) = \int \varphi(\gamma\tau)\psi(\tau^{-1})\, d\lambda^{s(\gamma)}(\tau) .$$

Together with the natural involution, this makes $\mathcal{K}(\mathcal{G})$ into a $*$-algebra. The latter can be $\mathbf{L}^1$ completed to a Banach $*$-algebra $\mathbf{L}^1(\mathcal{G})$. Its universal enveloping C*-algebra $\mathrm{C}^*(\mathcal{G})$ is the *groupoid C*-algebra* of $\mathcal{G}$.

2.1.3. Groupoids exist in abundance. The fundamental example is the *transformation groupoid* $\mathcal{G} = Y \rtimes G$ where Y is a locally compact space acted upon from the right by the locally compact group G. Set theoretically, it is the direct product $Y \times G$. The groupoid structure is given by

$$r(y,g) = y \in Y = \mathcal{G}^{(0)}\,,\ s(y,g) = y.g\,,\ (y,g) \circ (y.g,h) = (y,gh) \quad \text{and} \quad \lambda^y = \delta_y \otimes \mu$$

where μ is Haar measure on G. Special cases include $\mathcal{G} = Y$ $(G = 1)$ and $\mathcal{G} = G$ $(Y = \mathrm{pt})$.

If $U \subset \mathcal{G}^{(0)}$ is some locally closed subset, then $\mathcal{G}|U = r^{-1}(U) \cap s^{-1}(U)$, the set of arrows beginning and ending in U, is a locally compact space satisfying conditions (G1) and (G2). The circumstances under which condition (G3) holds for the restriction of λ are quite delicate in general. For the case $\mathcal{G} = Y \rtimes G$ of a transformation groupoid, is is necessary and sufficient that the following holds true [Nic87, prop. 1.3]:

(R1). For $y^{-1}U = \{g \in G | y.g \in U\}$ we have $\mu \mid y^{-1}U$ is fully supported on $y^{-1}U$ for all $y \in U$.

(R2). The map $U \to \mathbf{L}^\infty(G) : y \mapsto 1_{y^{-1}U}$ is weakly continuous, 1_A denoting the characteristic function of A.

2.1.4. With these definitions at hand, let us consider the following example. The vector space X acts on itself by translations. Let

$$\mathcal{G} = (X \rtimes X)|\Omega = \big\{(x,y) \in \Omega \times X \bigm| y \in \Omega - x\big\}\ .$$

This is a groupoid with Haar system $\lambda^x = \delta_x \otimes \lambda_X | (\Omega - x)$, λ_X denoting Lebesgue measure on X. We can define a $*$-representation of $\mathrm{C}^*(\mathcal{G})$ on $\mathbf{L}^2(\Omega)$ by

$$L(\varphi)u(x) = \int_\Omega \varphi(x, y-x)u(y)\,dy \quad \text{for all} \quad \varphi \in \mathcal{K}(\mathcal{G})\,,\ u \in \mathbf{L}^2(\Omega)\,,\ x \in \Omega\ .$$

It is easy to see that the image of L consists only of Wiener-Hopf operators. However, the $L(\varphi)$ are all compact, so not the entire Wiener-Hopf algebra can be realised in this fashion. In order to achieve this, a compactification of X has to be constructed which is equivariant for the action of the Abelian group X.

2.2 Wiener-Hopf compactification

2.2.1. Let $\mathbb{F}(X)$ be the set of all closed subsets of X. There exists a unique metric topology on X such that convergence is characterised by the equality $\underline{\lim}_k A_k = \overline{\lim}_k A_k$ where $\underline{\lim}_k A_k$ is the set of limits $a = \lim_k a_k$ for $a_k \in A_k$ and $\overline{\lim}_k A_k$ is the set of $a = \lim_k a_{\alpha(k)}$ for $a_{\alpha(k)} \in A_{\alpha(k)}$, $\alpha : \mathbb{N} \to \mathbb{N}$ denoting a subsequence of the identity. This convergence is called *Painlévé-Kuratowski convergence*. With this topology, $\mathbb{F}(X)$ is compact and separable.

The group X acts continuously on $\mathbb{F}(X)$, and the map $X \to \mathbb{F}(X) : x \mapsto x - \Omega$ is continuous, injective and equivariant. Identify X and Ω with their images under this embedding and denote their respective closures by $\overline{X}$ and $\overline{\Omega}$. It is a non-trivial fact due to Nica [Nic87,] that the restricted transformation groupoid $\mathcal{W}_\Omega = (\overline{X} \rtimes X)|\overline{\Omega}$ has the Haar system $\lambda^A = \delta_A \otimes \lambda_X|(A^{-1}\overline{\Omega})$. We call $\mathcal{W}_\Omega$ the *Wiener-Hopf groupoid.*

Moreover, we have the following theorem [Nic87, prop. 2.4.1], [HN95, th. III.14, th. IV.11].

Theorem 2.2.2. Let $\mathcal{W}_\Omega$ be the Wiener-Hopf groupoid.

(i). The groupoid C*-algebra C$^*(\mathcal{W}_\Omega)$ has a faithful $*$-representation L on $\mathbf{L}^2(\Omega)$, given by

$$L(\varphi)u(x) = \int_\Omega \varphi(x-\Omega, y-x)u(y)\,dy \quad \text{for all} \quad \varphi \in \mathcal{K}(\mathcal{W}_\Omega)\,,\ u \in \mathbf{L}^2(\Omega)\,,\ x \in \Omega\,.$$

Then for $\tilde{\varphi}(A,y) = \varphi(-y)$, $\varphi \in \mathcal{K}(X)$, we have $L(\tilde{\varphi}) = W_\varphi$.

(ii). The image of L is precisely the Wiener-Hopf algebra $\mathcal{W}(\Omega)$.

(iii). The Wiener-Hopf algebra $\mathcal{W}(\Omega)$ contains the compact operators $\mathbb{K}(\mathbf{L}^2(\Omega))$. Under L, this ideal corresponds to the subalgebra C$^*(\mathcal{W}_\Omega|\Omega)$ of C$^*(\mathcal{W}_\Omega)$ generated by the continuous functions φ with compact support in $\mathcal{W}_\Omega|\Omega = (X \rtimes X)|\Omega$.

2.2.3. The theorem contains the basic philosophy of the groupoid approach to the study of WH operators in a nutshell: The WH algebra is viewed as the set of functions on the non-commutative space (i.e., the groupoid) obtained by geometric compactification from Ω; its ideals correspond to certain *reductions* of this groupoid.

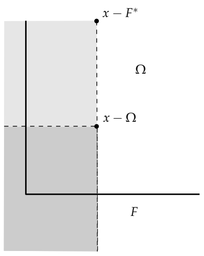
More precisely, this leads to the notion of *invariant subset* of the unit space $\mathcal{G}^{(0)}$ of a groupoid $\mathcal{G}$. A locally closed subset $U \subset \mathcal{G}^{(0)}$ is called (right) invariant, if $r(\gamma) \in U$ implies $s(\gamma) \in U$. It is clear that the restriction to $\mathcal{G}|U$ of a Haar system on $\mathcal{G}$ automatically satisfies the invariance and support conditions on a Haar system, if U is invariant. If U is, moreover, open, then this restriction is in fact a Haar system. It follows that in this case, C$^*(\mathcal{G}|U)$ is naturally an ideal of C$^*(\mathcal{G})$.

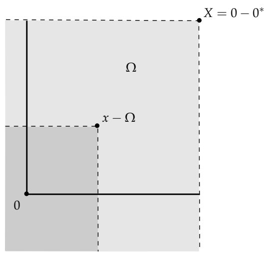
In the case of the Wiener-Hopf groupoid, $\Omega \subset \overline{\Omega} = \mathcal{W}_\Omega^{(0)}$ is an open invariant subset. This follows from the non-trivial fact that $\overline{\Omega}$ is a regular compactification of Ω [HN95, th. II.11]. Moreover, in this framework, the best possible description of a composition series would be to find a suitably fine filtration of $\overline{\Omega}$ by open invariant subsets. This requires a better understanding of the WH compactification.

2.3 Fine structure of the Wiener-Hopf compactification

2.3.1. It is easy to believe (though less easy to prove) that the description of $\overline{X}$ can be reduced to that of $\overline{\Omega}$; indeed, any non-void element of $\overline{X}$ lies on the X-orbit of some element of $\overline{\Omega}$ [HN95, lem. II.18]. Thus, we concentrate on the description of $\overline{\Omega}$.

Let us consider, as an example, the quarter plane $\Omega = [0,1[^2 \subset \mathbb{R}^2 = X$. This cone is self-dual and simplicial. Identifying a point $x \in \Omega$ with the set $x - \Omega$, we see that limits of sequences x_k can contribute to $\overline{\Omega} \setminus \Omega$ in two distinct fashions. Either, one of the components of x_k remains bounded; in this case, the limit point will be an affine half space not completely containing Ω. Or, both components tend to infinity; in which case, the limit shall be the entire space X. This is illustrated below.



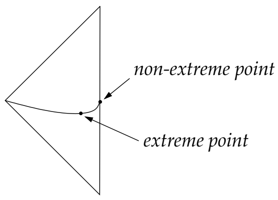


Recall that a *face* of Ω is a subcone of F such that for any segment $[x,y]\subset\Omega$ such that $]x,y[$ intersects F, we have $[x,y]\subset F$. Moreover, denote $C^*=\{x\in X|(x:C)\geqslant 0\}$ for any $C\subset X$. Then any point in $\overline{\Omega}$ is of the form $x-F^*$, where F is some face. This is illustrated above for $F=]0,\infty]\times 0$ and $F=0$. The points in Ω lie above $F=\Omega^*$. Passing from the example to the general case, we have the following theorem [Nic87, prop. 4.6.2].

Theorem 2.3.2. Let P be the of faces of Ω^*. Denote by $F^{\circledast}=F^*\cap\langle F\rangle$ the dual cone of F relative to its span $\langle F\rangle$. Then the following map is a well-defined injection,

$$\nu:\bigcup\nolimits_{F\in P}\{F\}\times F^{\circledast}\to\overline{\Omega}:(F,x)\mapsto x-F^*\ .$$

2.3.3. The point of the theorem is that the range of ν is in fact contained in $\overline{\Omega}$. The proof relies on the embedding of faces in chains of relatively exposed faces. Nica himself gives a counterexample for the surjectivity of ν, namely, the four-dimensional cone Ω^* with the following base.



It is evident that the set of extreme rays of Ω^* is non-compact. However, this is necessary for the surjectivity of Nica's map, as we shall see presently.

2.3.4. The set P is contained in $\mathbb{F}(X)$, but since the dual cone map $C\mapsto C^*$ is an homeomorphism on the subset of closed convex cones, P also carries the subspace topology under the embedding $P\to\overline{\Omega}:F\mapsto -F^*$. If $-F_k^*\to x-F^*$ where $F_k,F\in P$ and $x\in F^{\circledast}$, then $x=0$. Thus, if ν is surjective, then P is closed, and hence compact.

Let $\{n_0<\cdots<n_d\}=\{\dim F|F\in P\}$ be the set of face dimensions, ordered increasingly. Moreover, let $P_j=\{F\in P|\dim F=n_{d-j}\}$. The dimension function is easily seen to be lower semi-continuous, so $\bigcup_{i=j}^d P_i$ is closed in P. Thus, it seems reasonable to assume that all the P_j are compact. In fact, it turns out that this condition is sufficient for the surjectivity of ν.

Theorem 2.3.5. Define $\pi(x-F^*)=F$ for $F\in P$ and $x\in F^{\circledast}$.

(i). If P_j is compact, then $\pi:\pi^{-1}(P_j)\to P_j$ is continuous.

(ii). If P_j is compact, then it is a finite-dimensional metric space.

(iii). If all the P_j, $j=0,\ldots,d$, are compact, then ν is surjective.

2.3.6. The upshot of the theorem is that we can think of $\overline{\Omega}$ as stratified by fibre bundles, as soon as the sets P_j of faces of fixed dimension are compact.

Part (i) of the theorem is easy, and part (ii) follows by embedding P_j into a suitable Grassmannian variety. The proof of (iii) is lengthy, the essential step being the following technical statement: If $x_k - F_k^* \to C$ and $F_k \to F$ where $\dim F_k = \dim F$, then $\dim \operatorname{dom} \sigma_C < \dim F$ whenever (x_k) is unbounded. Here, $\sigma_C(x) = \sup_{y \in C}(x : y)$ is the support functional of C and $\operatorname{dom} \sigma_C$ is the set where it attains finite values.

It does not seem to be clear whether the condition that the P_j be compact is necessary for ν to be surjective. Clearly, the compactness of P does not imply that of the P_j, as can be seen by considering a three-dimensional cone whose compact base is the convex hull of the set of points on a circle whose angles belong to a Cantor set. Also, the facial exposedness of Ω^* is apparently unrelated to the compactness of the P_j condition. The usual 'parking ramp', which is not facially exposed, has P_j compact for all j.

On the positive side, polyhedral cones have compact P_j. Moreover, so do irreducible symmetric cones. The class of cones with compact face spaces is closed under finite products. Philosophically, all sensible cones, and some others, have compact face spaces P_j.

In the remainder of this paper, we shall always assume the P_j to be compact.

2.4 Construction of a composition series

2.4.1. The subsets $P_j \subset P$ naturally give rise to the open invariant subsets $U_j = \pi^{-1}\bigl(\bigcup_{i=1}^{j-1}\bigr)$, $j = 0, \dots, d+1$, of $\mathcal{W}_\Omega^{(0)} = \overline{\Omega}$. The fact that these subsets are indeed open follows directly from the lower semi-continuity of $\dim \circ \pi$. We point out that $U_0 = \varnothing$, $U_1 = \Omega$ and $U_{d+1} = \overline{\Omega}$.

Then the groupoids $\mathcal{W}_\Omega \,\big|\, U_j$ all have Haar systems. In fact, $I_j = \mathrm{C}^*(\mathcal{W}_\Omega \,\big|\, U_j)$ can be considered as a closed $*$-ideal of $\mathrm{C}^*(\mathcal{W}_\Omega)$. We have already noticed that I_1 corresponds to $\mathbb{K}(\mathbf{L}^2(\Omega))$ under the WH representation L from theorem 2.2.2. Let us determine the other ideals.

Set $Y_j = U_{j+1} \setminus U_j = \pi^{-1}(P_j)$, a locally closed invariant subset of $\mathcal{W}_\Omega^{(0)} = \overline{\Omega}$. Since Y_j is invariant, again $\mathcal{W}_\Omega \,\big|\, Y_j$ has a Haar system. Moreover, by [Ren80, ch. II, prop. 4.5], $I_{j+1}/I_j \cong \mathrm{C}^*(\mathcal{W}_\Omega \,\big|\, Y_j)$, where the isomorphism is induced by restriction $\varphi \mapsto \varphi \,\big|\, Y_j$ of compactly supported continuous functions φ on $\mathcal{W}_\Omega \,\big|\, U_{j+1}$.

We intend to show that the quotient I_{j+1}/I_j is stably isomorphic to a commutative C*-algebra. In order to do this, we show that the groupoid $\mathcal{W}_\Omega \,\big|\, Y_j$ is a fibre bundle.

Theorem 2.4.2. Let $\Sigma_j = \bigl\{(F, v) \,\big|\, F \in P_j\,,\, v \in F^\perp\bigr\}$ be the 'normal bundle' of P_j. Then Σ_j is a vector bundle over P_j, and $\mathcal{W}_\Omega|Y_j$ is a locally trivial continuous family of groupoids over Σ_j, with local trivialisations given by

$$\begin{aligned} &\mathcal{W}_\Omega \,\big|\, \pi^{-1}(U_E) \to \Sigma_j \,\big|\, U_E \times \mathcal{W}_{E^\circledast} \,\big|\, E^\circledast : \\ &\qquad (F, x, y) \mapsto \bigl(F, p_{F^\perp}(y), \psi_{EF}(x), \psi_{EF}(p_{F^\perp}(y))\bigr)\,. \end{aligned} \tag{2.1}$$

Here $U_E \subset P_j$ is an open neighbourhood of E, and $\psi_{EF} : \langle F \rangle \to \langle E \rangle$ is a bi-Lipschitz map which satisfies

$$\psi_{EF}(F^\circledast) = E^\circledast \quad \text{and} \quad \det \psi'_{EF}(x) = 1 \quad \text{for a.e.} \quad x\,.$$

Moreover, $C^*(\mathcal{W}_\Omega \mid Y_j)$ is thereby isomorphic to the section algebra $\mathbb{K}(\mathcal{E}_j)$ of the continuous field of elementary C*-algebras $C^*(\mathcal{W}_{E^\circledast} \mid E^\circledast) = \mathbb{K}(\mathbf{L}^2(E^\circledast))$ associated to the continuous field of Hilbert spaces $\mathcal{E}_j = \bigl(\mathbf{L}^2(F^\circledast)\bigr)_{(F,y)\in\Sigma_j}$.

Proof. The maps ψ_{EF} are constructed by deforming equal-dimensional cones into each other. Such a deformation can be realised by considering the Minkowski gauge functionals for compact bases of the corresponding cones.

The essential point is to compute the derivatives of these gauge functionals, showing that they have positive Jacobian. The property $\det \psi_{EF} = 1$ a.e. is tantamount in proving that C*-algebras of the obtained groupoids are isomorphic, since it shows that the Haar systems correspond. The proof that $C^*(\mathcal{W}_\Omega \mid Y_j)$ equals $\mathbb{K}(\mathcal{E}_j)$ is now standard. ☐

2.4.3. Since Σ_j is a finite-dimensional metric space, the field $\mathcal{E}_j$ is trivial. Hence, $\mathbb{K}(\mathcal{E}_d) = \mathcal{C}_0(X)$, and $\mathbb{K}(\mathcal{E}_j) = \mathcal{C}_0(\Sigma_j) \otimes \mathbb{K}$ if $j < d$. Thus, we have the following theorem.

Theorem 2.4.4. The I_j form an ascending chain of ideals of $C^*(\mathcal{W}_\Omega)$. We have

$$I_{j+1}/I_j \cong \begin{cases} \mathcal{C}_0(X) & j = d \ , \\ \mathcal{C}_0(\Sigma_j) \otimes \mathbb{K} & 0 \leqslant j < d \ . \end{cases}$$

2.4.5. We point out that the above isomorphisms are realised by the following representation $\sigma_j = (L^{F,y})_{(F,y)\in\Sigma_j}$ of $C^*(\mathcal{W}_\Omega)$ on $\mathcal{E}_j$,

$$L^{F,y}(\varphi)h(v) = \int_{F^\perp} \int_{F^\circledast} \varphi(F, v, w_1 + w_2 - v) e^{-i(y:w_2)} h(w_1)\, dw_1\, dw_2$$

for all $\varphi \in \mathcal{K}(\mathcal{W}_\Omega)$, $(F,y) \in \Sigma_j$, $h \in \mathbf{L}^2(F^\circledast)$, $v \in F^\circledast$.

Let us briefly review our theorem in the case of the quarter plane $\Omega = \Omega^* = [0,\infty[^2$ considered in 2.3.1. We have $d = 2$,

$$P_0 = \{\Omega^*\} \ , \ P_1 = \bigl\{[0,\infty[\times 0, 0 \times [0,\infty[\bigr\} \quad \text{and} \quad P_2 = \{0\} \ .$$

Thus,

$$\Sigma_0 = \mathrm{pt} \ , \ \Sigma_1 = 2 \cdot \mathbb{R} \quad \text{and} \quad \Sigma_2 = X = \mathbb{R}^2 \ ,$$

where $2 \cdot \mathbb{R}$ denotes the topological sum of 2 copies of $\mathbb{R}$. The theorem gives

$$I_0 = 0 \ , \ I_1 = \mathbb{K} \ , \ I_2/I_1 \cong \mathcal{C}_0(\mathbb{R}) \otimes \mathbb{C}^2 \otimes \mathbb{K} \quad \text{and} \quad I_3/I_2 = \mathcal{C}_0(\mathbb{R}^2) \ .$$

Here, $\mathbb{C}^2$ is the commutative C*-algebra of functions on 2 points.

More generally, for the simplicial cone $\Omega = \Omega^* = [0,\infty[^n$, we get

$$d = n \ , \ \Sigma_j = \tbinom{n}{j} \cdot \mathbb{R}^j \quad \text{and} \quad I_{j+1}/I_j = \mathcal{C}_0(\mathbb{R}^j) \otimes \mathbb{C}^{\binom{n}{j}} \otimes \mathbb{K} \quad \text{for all} \ \ j < n \ .$$

Of course, these results also follow from the work of Muhly-Renault [MR82] on polyhedral cones.

3 Index Theory

3.1 Construction of an Analytical Index

3.1.1. The composition series constructed in theorem 2.4.4 gives rise to commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & I_j/I_{j-1} \arrow[r] \arrow[d, equal, "\sigma_{j-1}"'] & I_{j+1}/I_{j-1} \arrow[r, "\sigma_j"] \arrow[d, "\sigma_{j-1}"'] & \mathbb{K}(\mathcal{E}_j) \arrow[r] \arrow[d, dashed, "\tau_j"] & 0 \\ 0 \arrow[r] & \mathbb{K}(\mathcal{E}_{j-1}) \arrow[r] & \mathcal{L}(\mathcal{E}_{j-1}) \arrow[r, "\pi_{j-1}"'] & \mathcal{L}/\mathbb{K}(\mathcal{E}_{j-1}) \arrow[r] & 0 \end{tikzcd}$$

Here, $\mathcal{L}(\mathcal{E}_{j-1}) = \mathrm{M}(\mathbb{K}(\mathcal{E}_{j-1}))$ is the C^*-algebra of adjointable operators of the Hilbert $\mathcal{C}_0(\Sigma_{j-1})$-module $\mathcal{E}_{j-1}$. We point out that I_j is an essential ideal of I_{j+1}, so that I_{j+1} injects into $\mathrm{M}(I_j)$, and thus σ_{j-1}, which coincides with the strict extension of its restriction to I_j, takes values in $\mathcal{L}(\mathcal{E}_{j-1})$ when evaluated on I_{j+1}.

The map τ_j is the *Busby invariant* of the extension I_{j+1}/I_{j-1} of I_j/I_{j-1} by $\mathbb{K}(\mathcal{E}_j)$. It may be computed as $\pi_{j-1}\circ\sigma_{j-1}\circ\varrho_j$, whenever ϱ_j is a completely positive section of σ_j. Such ϱ_j exist since I_{j+1}/I_{j-1} is nuclear, being of type I.

Any short exact sequence $0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$ of C^*-algebras defines, by the six-term exact sequence in K-theory, a map $\partial : K_1(B) \to K_0(J)$, only depending on π. A simple description in terms of E-theory can be given as follows, cf. [Con94, II.B.γ, lem. 6]. Let $0 \leqslant u_t \leqslant 1$, $t \in [1,\infty[$, be a continuous approximate unit of J, quasi-central for A, and $\varrho : B \to A$ a section of π (not necessarily completely positive). Then

$$\varphi_t(f\otimes b) = f(u_t)\varrho(b) \quad \text{for all} \quad f \in \mathcal{C}_0(]0,1[)\,,\, b \in B\,,\, t \in [1,\infty[\ ,$$

defines an asymptotic $*$-morphism from $SB = \mathcal{C}_0(]0,1[)\otimes B$ to J. The map associated to (φ_t) in K-theory $K_1(B) = K_0(SB) \to K_0(J)$ (given by composition in E-theory) coincides with ∂.

Denote the map $K_1(\mathbb{K}(\mathcal{E}_J)) \to K_0(\mathbb{K}(\mathcal{E}_{j-1}))$ given by this construction by Ind_j. Because of the naturality of connecting homomorphisms,

$$\mathrm{Ind}_j([u]) = \partial[\tau_j(u)] \quad \text{for all} \quad [u] \in K_1(\mathbb{K}(\mathcal{E}_j)) = \pi_0\big(\mathrm{GL}_\infty(\mathbb{K}(\mathcal{E}_j)^+)\big)$$

where $\partial : K_1(\mathcal{L}/\mathbb{K}(\mathcal{E}_{j-1})) \to K_0(\mathbb{K}(\mathcal{E}_{j-1}))$ is the connecting homomorphism.

3.1.2. Call an element $a \in M_N(\mathrm{C}^*(\mathcal{W}_\Omega))$ *j-Fredholm*, if there exists some $b \in M_N(\mathrm{C}^*(\mathcal{W}_\Omega))$, such that $ab - 1 \equiv ba - 1 \equiv 0 \pmod{M_N(I_J)}$. Then any K-theory representative $u \in \mathrm{GL}_N(\mathbb{K}(\mathcal{E}_j))$ lifts to a j-Fredholm element $\varrho_j(u)$. It can be shown that $\sigma_{j-1}(\varrho_j(u))$ is a continuous family of $N\times N$ Fredholm matrices on the continuous family $\mathcal{E}_{j-1}$ of Hilbert spaces. Using the identification $K_1(\mathbb{K}(\mathcal{E}_j)) = K^1_c(\Sigma_j)$ (topological K-theory with compact supports), we can, moreover, prove that the family $\sigma_{j-1}(\varrho_j(u))$ is trivial outside a compact subset of Σ_{j-1}. Thus, the Atiyah-Jänich family index $\mathrm{Index}_{\Sigma_{j-1}}\sigma_{j-1}(\varrho_j(u))$ makes sense as an element of $K^0_c(\Sigma_{j-1})$. We have the following theorem.

Theorem 3.1.3. Let $[u] \in K^1_c(\Sigma_j) = K_1(\mathbb{K}(\mathcal{E}_j))$. Then

$$\mathrm{Ind}_j([u]) = \mathrm{Index}_{\Sigma_{j-1}}\sigma_{j-1}(\varrho_j(u)) \in K^0_c(\Sigma_{j-1}) = K_0(\mathbb{K}(\mathcal{E}_{j-1}))\ .$$

3.2 Construction of a Topological Index

3.2.1. The next step in proving an index theorem is the definition of a topological index. To that end, note that $\mathcal{P}_j = \{(E,F) \in P_{j-1} \times P_j | E \supset F\}$ is a compact space in the topology induced by $P_{j-1} \times P_j$. The projections $P_{j-1} \xleftarrow{\xi} \mathcal{P}_j \xrightarrow{\eta} P_j$ turn $\mathcal{P}_j$ into a fibre bundle over their respective compact images (which need not be all of P_{j-1} resp. P_j, lest the face lattice P be modular). Since the natural map $\eta^*\Sigma_j \to \Sigma_j$ is proper, we get a map in K-theory,

$$\eta^* : K_c^1(\Sigma_j) \to K_c^1(\eta^*\Sigma_j) = K_c^0(\eta^*\Sigma_j \times \mathbb{R}) \ .$$

Each of the fibres $\xi^{-1}(E)$ for $E \in \xi(\mathcal{P}_j)$ has a natural Euclidean embedding, as follows.

Proposition 3.2.2. Let $(E,F) \in \mathcal{P}_j$. Write $E_0(F) = \langle F \rangle$, $E_1(F) = \langle F^\perp \cap E^\circledast \rangle$, and

$$E_{1/2}(F) = E_0(F)^\perp \cap E_1(F)^\perp \cap \langle E \rangle \ .$$

We have

(i). $E_1(F) = \mathbb{R} \cdot e_F$ for a unique $e_F \in E^\circledast$, $\|e_F\| = 1$.

(ii). The set $S_1(E) = \{e_F | F \in P_j \ , \ F \subset E\} \subset$ is a compact $\mathcal{C}^{(1)}$-submanifold of X, and its tangent space at e_F is $E_{1/2}(F)$.

(iii). The map $\xi^{-1}(E) \to S_1(E) : (E,F) \mapsto e_F$, is an homeomorphism.

3.2.3. Thus, it is natural to define $T\mathcal{P}_j = \big\{(E,F,u) \bigm| (E,F) \in \mathcal{P}_j \ , \ u \in E_{1/2}(F)\big\}$, the tangent space of $\mathcal{P}_j$ along the fibres of ξ. Moreover, consider $\varrho : \Sigma_{j-1} \to P_{j-1} : (E,v) \mapsto E$. The 1-dimensionality of $E_1(F)$ implies that the map

$$\eta^*\Sigma_j \to \varrho^* T\mathcal{P}_j : (E,F,y) \mapsto \big(E, p_{E^\perp}(y), F, p_{E_{1/2}(F)}(y)\big)$$

turns $\eta^*\Sigma_j$ to a real line bundle over $\varrho^*T\mathcal{P}_j$. Here, we note $F^\perp = E^\perp \oplus E_{1/2}(F) \oplus E_1(F)$. Thus, $\eta^*\Sigma_j \times \mathbb{R}$ is a complex line bundle over $\varrho^*T\mathcal{P}_j$, and we get the corresponding Thom isomorphism $\beta : K_c^0(\varrho^*T\mathcal{P}_j) \to K_c^0(\eta^*\Sigma_j \times \mathbb{R}) = K_c^1(\eta^*\Sigma_j)$.

3.2.4. In order to construct an index map $K_c^0(\varrho^*T\mathcal{P}_j) \to K_c^0(\Sigma_{j-1})$, we use the device of the (fibre-wise) *tangent groupoid*, due to Connes [Con94, II.5], and a certain C*-algebraic deformation quantisation. Let $\xi : M \to B$ be a fibre bundle with C^1 fibres (the C^1 structure depending continuously on the base points), and let $TM = \bigcup_{b \in B} \{b\} \times T\xi^{-1}(b)$ be the fibrewise tangent bundle. Then a groupoid $\mathcal{G}_M$, called the fibrewise tangent groupoid, can be constructed in the following manner.

Set-theoretically, let $\mathcal{G}_M = TM \times 0 \cup (M \times_B M) \times]0,1]$. Endow this space with the topology generated by $(M \times_B M) \times]0,1]$ and the maps $\phi_h : \mathcal{G}_M \to M \times \mathbb{R} \times [0,1]$ where

$$\phi_h(x,X,0) = (x, dh_x(X), 0) \quad \text{and} \quad \phi_h(x_1,x_2,\varepsilon) = \left(x_1, \frac{h(x_1) - h(x_2)}{\varepsilon}, \varepsilon\right) \ ,$$

and $h : M \to \mathbb{R}$ is any continuous map which is $\mathcal{C}^1$ along the fibres of ξ. Then $\mathcal{G}_M$ becomes a locally compact groupoid with unit space $\mathcal{G}_M^{(0)} = M \times [0,1]$ by considering TM as a 'group bundle' over M, and $M \times_B M$ as a subgroupoid of the pair groupoid $M \times M$. A more delicate matter is the existence of a Haar system on $\mathcal{G}_M$.

To that end, it is useful to consider $\mathcal{G}_M$ as a continuous family groupoid (of class $\mathcal{C}^{1,0}$) in the sense of Paterson [Pat00, § 3, def. 3]. I.e., we need to show that $r, s : \mathcal{G}_M \to \mathcal{G}_M^{(0)}$

turn $\mathcal{G}_M$ into a continuous family of $\mathcal{C}^1$ manifolds, and that inversion and composition are compatible with this structure. We indicate how to define an atlas.

Whenever (α, U_α) is a $\mathcal{C}^{1,0}$ atlas of $\xi : M \to B$, let $V_\alpha = \mathcal{G}_M \mid (U_\alpha \times [0,1])$, and define a chart $\varphi_\alpha : V_\alpha \to U_\alpha \times [0,1] \times \mathbb{R}^n$ where $n = \dim \xi^{-1}(b)$, by letting the components of φ_α in $\mathbb{R}^n$ be $\phi_{\alpha_1}, \dots, \phi_{\alpha_n}$. A routine proof shows that this defines an atlas with the required properties, and we may apply [Pat00, § 3, th. 1] to see that $\mathcal{G}_M$ has a Haar system which is unique up to multiplication with positive $\mathcal{C}^{1,0}$ densities. (Note that although Paterson states his result for $\mathcal{C}^{\infty,0}$ groupoids, only $\mathcal{C}^{1,0}$ structure is needed.)

3.2.5. The construction set forth in the previous paragraph gives rise to a continuous family of groupoid C*-algebras, as follows. Let $A = \mathrm{C}^*(\mathcal{G}_M)$, and consider $p : \mathcal{G}_M \to [0,1]$, the projection onto the deformation parameter. Since p is open, factors through $\mathcal{G}_M^{(0)}$, and its fibres $\mathcal{G}_M^t = p^{-1}(t)$ are amenable groupoids, the groupoid $\mathcal{G}_M$ is amenable, too [ADR00, prop. 5.3.4]. Moreover, by the work of Landsman-Ramazan [LR01, § 5], $A_t = \mathrm{C}^*(\mathcal{G}_M^t)$ is a continuous family of C*-algebras, whose associated C*-algebra is just A.

We have evaluation maps $\varepsilon_t : A \to A_t$ for all $t \in [0,1]$, and a standard argument [Con94, II.5, prop. 5] shows that ε_0 induces an isomorphism in K-theory. Thus, we may define

$$\text{q-ind} = \varepsilon_1 \varepsilon_0^{-1} : K_c^0(TM) = K_0(\mathrm{C}^*(\mathcal{G}_M^0)) \to K_0(\mathrm{C}^*(\mathcal{G}_M^1)) = K_c^0(B) \ .$$

Here, the identifications with the topological K-groups arise on one hand by applying the Fourier transform fibrewise, and on the other hand by noting that $\mathrm{C}^*(M \times_B M)$ is the C*-algebra associated to a continuous field of *elementary* C*-algebras on B which is trivial as soon as B is finite-dimensional as a topological space. This is the desired index map.

We point out that there is a standard procedure, also due to to Connes, to compute such an index map in topological terms, which then allows for a cohomological index formula by application of the Chern character. We refer the reader to [Con94, II.5, pp. 104/5], [Lan03, § 5] for details. Generally speaking, the classifying space $B\mathcal{G}_M$ of $\mathcal{G}_M$ has to be computed as the quotient of a principal $\mathcal{G}_M$-space $E\mathcal{G}_M$ with contractible fibres. In fact, $E\mathcal{G}_M = \mathcal{G}_M^{(0)} \times \mathbb{R}^{2n}$ where some Euclidean embedding $M \hookrightarrow B \times \mathbb{R}^n$ is chosen. Then q-ind is, up to a Thom isomorphism, the Gysin map $i_!$ associated to the embedding $i : NM \hookrightarrow B \times \mathbb{R}^{2n}$, where NM is the fibrewise normal bundle of M in $B \times \mathbb{R}^{2n}$.

3.2.6. We may now consider the $\mathcal{C}^{1,0}$ fibre bundle $\varrho^*\mathcal{P}_j \to \xi(\mathcal{P}_j) \subset \Sigma_{j-1}$, where $\varrho : \Sigma_{j-1} \to P_{j-1}$ and $\xi : \mathcal{P}_j \to P_{j-1}$ are the natural projections. By the above procedure, we obtain an index map $\text{q-ind}_j : K_c^0(\varrho^* T\mathcal{P}_j) \to K_c^0(\xi(\mathcal{P}_j)) \to K_c^0(\Sigma_{j-1})$. Our 'topological index map' is thus

$$\text{q-Ind}_j = \text{q-ind}_j \circ \beta^{-1} \circ \eta^* : K_c^1(\Sigma_j) \to K_c^1(\Sigma_{j-1}) \ .$$

To complete our programme, we need to prove that $\mathrm{Ind}_j = \text{q-Ind}_j$. So far, this is open, although promising. We shall report in detail on our results in this direction to a later date.

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Q_p-spaces on bounded symmetric domains

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Abstract

Q_p spaces on the unit disc were introduced, and their basic properties established, in 1995 by Aulaskari, Xiao and Zhao. Later some of these results were extended also to the unit ball or even to strictly pseudoconvex domains in the complex n-space. We briefly review the theory of bounded symmetric domains, of which the disc and the ball are the simplest examples, and then discuss the Q_p spaces in this setting. It turns out that some new phenomena appear, most notably concerning the relationships of these spaces to the various kinds of Bloch spaces on symmetric domains.

1 Introduction

The Q_p spaces on the unit disc $\mathbf{D}$ were introduced in 1995 by Aulaskari, Xiao and Zhao [AXZ] by

$$f \in Q_p \iff \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 g(z,a)^p \, dz < \infty,$$

the square root of the right-hand side being, by definition, the (semi)norm in Q_p. Here $g(z,a)$ stands for the Green function

$$g(z,a) = \log\left|\frac{z-a}{1-\overline{a}z}\right|,$$

and dz denotes the Lebesgue area measure. It is not difficult to see that one gets the same spaces, with equivalent seminorms, upon replacing the Green function by the function $\log|\frac{z-a}{1-\overline{a}z}|$ which has (for each fixed a) the same boundary behaviour:

$$f \in Q_p \iff \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 \left(1 - \left|\frac{z-a}{1-\overline{a}z}\right|^2\right)^p dz < \infty. \tag{1.1}$$

We will adhere to this latter definition throughout the sequel.

The most notable feature of the Q_p spaces is that they are Möbius invariant. Indeed, any Möbius map (i.e. a biholomorphic self-map of $\mathbf{D}$) is of the form $\phi(z) = \epsilon\dfrac{a-z}{1-\overline{a}z}$, with $|\epsilon| = 1$ and $a \in \mathbf{D}$. Thus the right-hand side of (1.1) can be rewritten as

$$\begin{aligned}
&\sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} |f'(z)|^2 \; (1-|\phi(z)|^2)^p \, dz \\
&\qquad = \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \Delta|f|^2(z) \; (1-|\phi(z)|^2)^p \, dz \\
&\qquad = \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} (\widetilde{\Delta}|f|^2)(z) \; (1-|\phi(z)|^2)^p \, d\mu(z) \\
&\qquad = \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \widetilde{\Delta}|f\circ\phi(z)|^2 \; (1-|z|^2)^p \, d\mu(z),
\end{aligned}$$

where $\widetilde{\Delta} = (1-|z|^2)^2 \dfrac{\partial^2}{\partial z \partial \overline{z}}$ and $d\mu(z) = \dfrac{dz}{(1-|z|^2)^2}$ are the invariant Laplacian and the invariant measure on $\mathbf{D}$, respectively. From the last formula it is apparent that $f \in Q_p$ implies $f \circ \phi \in Q_p$ and f and $f \circ \phi$ have the same norm in Q_p, for all $\phi \in \mathrm{Aut}(\mathbf{D})$.

It was shown in [AXZ] that

$$\begin{aligned}
p > 1 \qquad &\Longrightarrow \qquad Q_p = \mathcal{B}, \quad \text{the Bloch space,} \\
p = 1 \qquad &\Longrightarrow \qquad Q_p = BMOA, \\
0 \le p_1 < p_2 \le 1 \qquad &\Longrightarrow \qquad Q_{p_1} \subsetneq Q_{p_2}, \\
p = 0 \qquad &\Longrightarrow \qquad Q_p = \mathcal{D}, \quad \text{the Dirichlet space,} \\
p < 0 \qquad &\Longrightarrow \qquad Q_p = \{\text{const}\}.
\end{aligned}$$

Thus the Q_p spaces provide a whole range of Möbius-invariant function spaces on $\mathbf{D}$ lying strictly between the Dirichlet space on the one hand, and $BMOA$ and the Bloch space on the other.

The Q_p spaces subsequently attracted a lot of attention; see e.g. the book by Xiao [X] and the references therein. They were generalized to the unit ball $\mathbf{B}^d \subset \mathbf{C}^d$ in 1998 by Ouyang, Yang and Zhao [OYZ]:

$$\begin{aligned}
f \in Q_p \iff & \sup_{a \in \mathbf{B}^d} \int_{\mathbf{B}^d} \widetilde{\Delta}|f(z)|^2 \, G(z,a)^p \, d\mu(z) < \infty \\
\iff & \sup_{\phi \in \mathrm{Aut}(\mathbf{B}^d)} \int_{\mathbf{B}^d} \widetilde{\Delta}|f \circ \phi|^2 \, G(z,0)^p \, d\mu(z) < \infty,
\end{aligned}$$

where $\widetilde{\Delta}, d\mu$ and $G(z,a)$ denote the invariant Laplacian, the invariant measure and the Green function of $\widetilde{\Delta}$ on $\mathbf{B}^d$, respectively. Again, these spaces are Möbius invariant, and

$$\begin{aligned}
p \ge \frac{d}{d-1}, \qquad &\Longrightarrow \qquad Q_p = \{\text{const}\}, \\
1 < p < \frac{d}{d-1} \qquad &\Longrightarrow \qquad Q_p = \mathcal{B}(\mathbf{B}^d), \quad \text{the Bloch space,} \\
p = 1 \qquad &\Longrightarrow \qquad Q_p = BMOA(\mathbf{B}^d), \\
\frac{d-1}{d} < p_1 < p_2 \le 1 \qquad &\Longrightarrow \qquad Q_{p_1} \subsetneq Q_{p_2}, \\
p \le \frac{d-1}{d} \qquad &\Longrightarrow \qquad Q_p = \{\text{const}\}.
\end{aligned}$$

The cut-off at $p = \frac{d}{d-1}$ turns out to due to the pole of $G(z,a)$ at $z = a$, and disappears if we replace (as we did for the disc) the Green function by $(1 - \|\frac{z-a}{1-\langle a,z\rangle}\|^2)^d$, i.e. upon setting

$$f \in Q_p \iff \sup_{\phi \in \mathrm{Aut}(\mathbf{B}^d)} \int_{\mathbf{B}^d} \widetilde{\Delta}|f \circ \phi|^2(z) \, (1-\|z\|^2)^{pd} \, d\mu(z) < \infty.$$

Then $Q_p = \mathcal{B}(\mathbf{B}^d) \ \forall p > 1$, while the other cases remain unchanged. (We again stick to this latter definition in the sequel.)

Note that, in contrast to the disc, for $d > 1$ the Dirichlet space does not turn up as one of the Q_p's, though in all other cases the situation is the same as for $\mathbf{D}$.

Other generalizations include Q_p spaces on smoothly bounded strictly pseudoconvex domains [AC] or the $F(p,q,s)$ spaces of Rättyä and Zhao [R],[Z]. In this talk, we will consider generalization in another direction, suggested by the appearance of the invariant Laplacians $\widetilde{\Delta}$, the invariant measures $d\mu$, and the invariance of the spaces under Möbius maps — namely, the generalization to bounded symmetric domains.

2 Bounded symmetric domains

Recall that a bounded domain $\Omega \subset \mathbf{C}^d$ is called *symmetric* if $\forall x \in \Omega$ there exists $s_x \in \mathrm{Aut}(\Omega)$ such that $s_x \circ s_x = \mathrm{id}$ and x is an isolated fixed-point of s_x. One calls s_x the geodesic symmetry at x. The motivating example behind this is, of course, the complex n-space $\Omega = \mathbf{C}^n$ with $s_x(z) = 2x - z$ (except that this is not a bounded domain). Another example is the unit disc $\mathbf{D}$ with $s_0(z) = -z$ and $s_x = \phi_x \circ s_0 \circ \phi_x$, where $\phi_x(z) = \dfrac{x-z}{1-\overline{x}z}$ is the geodesic symmetry interchanging 0 and x. A more general example is the unit ball $I_{r\times R}$ of $r \times R$ complex matrices ($R, r \geq 1$), again with $s_0(z) = -z$ and $s_x = \phi_x \circ s_0 \circ \phi_x$, the geodesic symmetry ϕ_x interchanging 0 and x being now given by

$$s_x(z) = (I_r - xx^*)^{-1/2}(x-z)(I_R - x^*z)^{-1}(I_R - x^*x)^{1/2}.$$

Note that this includes the unit ball $\mathbf{B}^d \subset \mathbf{C}^d$ as the special case I_{1d}.

It turns out that symmetry implies homogeneity: the group $\mathrm{Aut}(\Omega) := G$ acts transitively on Ω. (In fact, already the symmetries s_x do.) It is a semisimple Lie group.

A bounded symmetric domain is called irreducible if it is not biholomorphic to a Cartesian product of two other bounded symmetric domains.

Irreducible bounded symmetric domains were completely classified by E. Cartan. There are four infinite series of such domains plus two exceptional domains in $\mathbf{C}^{16}$ and $\mathbf{C}^{27}$:

Domain	Description	
I_{rR}	$Z \in \mathbf{C}^{r\times R}$: $\lVert Z\rVert_{\mathbf{C}^R\to\mathbf{C}^r} < 1$	$R \geq r \geq 1$
II_r	$Z \in I_{rr}$, $Z = Z^t$	$r \geq 2$
III_m	$Z \in I_{mm}$, $Z = -Z^t$	$m \geq 5$
IV_n	$Z \in \mathbf{C}^{n\times 1}$, $\lvert Z^tZ\rvert < 1$, $1 + \lvert Z^tZ\rvert^2 - 2Z^*Z > 0$	$n \geq 5$
V	$Z \in \mathbf{O}^{1\times 2}$, $\lVert Z\rVert < 1$	
VI	$Z \in \mathbf{O}^{3\times 3}$, $Z = Z^*$, $\lVert Z\rVert < 1$	

The restrictions on R, r, m, n stem from a few isomorphisms in low dimensions:

$$IV_1 \cong III_2 \cong II_1 \cong I_{1,1}(\cong \mathbf{D}), \quad IV_3 \cong II_2, \quad IV_2 \cong \mathbf{D}\times\mathbf{D},$$
$$IV_4 \cong I_{2,2}, \quad III_3 \cong I_{1,3}, \quad III_4 \cong IV_6, \quad I_{rR} \cong I_{Rr}.$$

Up to biholomorphic equivalence, any irreducible bounded symmetric domain is uniquely determined by three integers, namely its *rank* r and its *characteristic multiplicities* a, b.

domain	r	a	b	d	p
I_{rR} $(r \le R)$	r	2	$R-r$	rR	$r+R$
II_r	r	1	0	$\frac{1}{2}r(r+1)$	$r+1$
$III_{2r+\epsilon}$, $\epsilon \in \{0,1\}$	r	4	2ϵ	$r(2r+2\epsilon-1)$	$4r+2\epsilon-2$
IV_n	2	$n-2$	0	n	n
V	2	6	4	16	12
VI	3	8	0	27	18

The two other important quantities given in the table, the *genus* p and the dimension d are related to a, b and r by

$$p = (r-1)a + b + 2, \qquad d = \frac{r(r-1)}{2}a + rb + r.$$

The domains with $b = 0$ are in some respects "simpler" than others and are called *tube* domains. Thus, for instance, I_{rR} is tube $\iff$ $r = R$.

The unit balls $\mathbf{B}^d = I_{1d}$ are the only bounded symmetric domains of rank 1, and the only bounded symmetric domains with smooth boundary.

The domains in the list above are called *Cartan domains*. Clearly, any Cartan domain is convex, contains the origin, and is circular with respect to it.

From now on, we will suppose (unless explicitly stated otherwise) that Ω is a Cartan domain, and for each $x \in \Omega$ we denote by ϕ_x the (unique) geodesic symmetry interchanging 0 and x. We further denote by K the stabilizer of the origin in G,

$$K := \{k \in G : \; k0 = 0\}.$$

It is a consequence of one of Cartan's theorems that any $k \in K$ is automatically a unitary linear map on $\mathbf{C}^d$.

Note that from the definition of K it is immediate that any $\phi \in G$ is of the form $\phi = \phi_x k$, where $k \in K$, $x \in \Omega$. (In fact $x = \phi(0)$.)

Having recalled the definition of bounded symmetric domains, we can turn to the Q_p spaces on them. We have seen in the Introduction that their definition involves three ingredients — namely, the invariant Laplacian $\widetilde{\Delta}$, the invariant measure $d\mu$, and powers of the function $1 - |z|^2$ (or, for the ball, $1 - \|z\|^2$). Let us now clarify what are the counterparts of these on a general Cartan domain.

3 Invariant differential operators

A differential operator L on a Cartan domain Ω is called invariant if

$$L(f \circ \phi) = (Lf) \circ \phi \qquad \forall \phi \in G = \mathrm{Aut}(\Omega).$$

It is well known that on the unit disc, invariant operators are precisely the polynomials of the invariant Laplacian $\widetilde{\Delta} = (1-|z|^2)^2 \Delta$. The same is true for $\mathbf{B}^d$. For a general bounded symmetric domain, the situation is more complicated: namely, the algebra of all invariant differential operators consists of all polynomials in r commuting differential operators $\Delta_1, \dots, \Delta_r$, of orders $2, 4, \dots, 2r$, respectively, where r is the rank. In particular, the monomials $\Delta_1^{n_1} \dots \Delta_r^{n_r}$ form a linear basis of all invariant differential operators. However, often it is much more convenient to use another basis, the construction of which we now describe.

For any invariant differential operator L, let L_0 be the (non-invariant) linear differential operator obtain upon freezing the coefficients of L at the origin, that is, $Lf(0) =: L_0 f(0)$. From the invariance of L it follows that

$$k \in G,\ k0 = 0 \implies L_0(f \circ k) = (L_0 f) \circ k$$

(i.e. L_0 is K-invariant) and

$$Lf(z) = L_0(f \circ \phi_z)(0). \tag{3.1}$$

Conversely, if L_0 is a K-invariant constant-coefficient differential operator, then the recipe (3.1) clearly defines an invariant differential operator L on Ω. Thus there is a 1-to-1 correspondence between (G-)invariant linear differential operators on Ω and K-invariant linear constant-coefficients differential operators on $\mathbf{C}^d$.

Further, any constant-coefficient linear differential operator L_0 can be written in the form $L_0 = p(\partial, \overline{\partial})$ for some polynomial p on $\mathbf{C}^d \times \mathbf{C}^d$. It is not difficult to see that such operator is K-invariant if and only if the polynomial p is K-invariant in the sense that $p(x, \overline{y}) = p(kx, \overline{ky})\ \forall x, y \in \mathbf{C}^d\ \forall k \in K$.

Combining this with the observation in the preceding paragraph, we thus see that invariant differential operators are in 1-to-1 correspondence with K-invariant polynomials.

Example. Since K consists of unitary maps, the simplest K-invariant polynomial (apart from the constants) is $p(x, \overline{y}) = \langle x, y \rangle$. The corresponding invariant differential operator is

$$Lf(a) = \Delta(f \circ \phi_a)(0).$$

This operator is called the invariant Laplacian of Ω; it coincides with the Laplace-Beltrami operator with respect to the Bergman metric on Ω. Note that for f holomorphic,

$$L|f|^2(a) = \sum_{j=1}^{d} \Big| \frac{\partial (f \circ \phi_a)(0)}{\partial z_j} \Big|^2 = \| \partial (f \circ \phi_a)(0) \|^2$$

is what we might call the *invariant holomorphic gradient* of f.

In a moment, we will see that there exists a very natural basis for K-invariant polynomials (which will thus yield the sought basis for invariant differential operators). Prior to that, however, we review some facts about Bergman kernels on Cartan domains.

4 Bergman spaces and kernels

The function $h(x, y) := 1 - x\overline{y}$ on $\mathbf{D}$ is noteworthy in a number of ways. First of all, h^{-2} is, up to a constant factor, the Bergman kernel $K(x, y) = \dfrac{1}{\pi(1 - x\overline{y})^2}$. Second, $d\mu(z) = \dfrac{dz}{h(z, z)^2}$ is the invariant measure on $\mathbf{D}$. Finally, for any $\alpha > -1$, the Bergman kernel of $L^2_{\text{hol}}(\mathbf{D}, h(z, z)^\alpha\, dz)$ is given by

$$K_\alpha(x, y) = \frac{\text{const}}{h(x, y)^{\alpha+2}}.$$

The same properties are also possessed by the function $h(x,y) = 1 - \langle x, y\rangle$ on the ball, only in all three formulas 2 must be replaced by $d+1$.

It is a notable fact that the same phenomenon persists for a general Cartan domain. Namely, the Bergman kernel of a Cartan domain has the form

$$K(x,y) = \frac{1}{\operatorname{vol}\Omega}\, h(x,y)^{-p},$$

where $h(x,y)$ is an irreducible polynomial, analytic in x and $\overline{y}$, and such that $h(0,z) = h(z,0) = 1 \geq h(z,z) \geq 0$ $\forall z \in \Omega$. The degree of h is equal to the rank, r, and p is the genus (this is always an integer ≥ 2). Finally, the Bergman kernel of $L^2_{\mathrm{hol}}(\Omega, h(z,z)^\alpha\, dz)$ equals

$$K_\alpha(x,y) = \frac{\text{const}}{h(x,y)^{\alpha+p}},$$

for any $\alpha > -1$, and

$$d\mu(z) := \frac{dz}{h(z,z)^p}$$

is an invariant volume element on Ω.

For the domains I and II in Cartan's list, h is given by $h(X,Y) = \det(I - XY^*)$; for domains of type III, the determinant gets replaced by the Pfaffian. Explicit formulas are known also for the types IV–VI.

Comparing all the facts above with the situations for the disc and the ball, we see that we should define the Q_p spaces on a Cartan domain for any $\nu \in \mathbf{R}$ and any invariant differential operator L as follows:

$$f \in Q_{\nu,L} \iff \sup_{\phi\in G} \int_\Omega L|f\circ\phi|^2(z)\; h(z,z)^\nu\; d\mu(z) < \infty. \tag{4.1}$$

(Here we have started using the subscript ν instead of p since the letter p is already reserved for the genus.) Clearly, this reduces to the original definitions for $\Omega = \mathbf{D}$ (or $\mathbf{B}^d$) and L the invariant Laplacian.

A small catch here is, however, that in order to have the square-root of the right-hand side for a seminorm, we need this right-hand side to be nonnegative for all holomorphic functions f. It is precisely at this point that the promised linear basis for invariant differential operators comes to the rescue; so let us exhibit it without further delay.

5 Peter-Weyl decomposition

Let $\mathcal{P}$ denote the vector space of all (holomorphic) polynomials on $\mathbf{C}^d$. We endow $\mathcal{P}$ with the Fock inner product

$$\begin{aligned}\langle f, g\rangle_F :&= f(\partial)\; g^*\,(0), \qquad \text{where} \quad g^*(z) := \overline{g(\overline{z})}, \\ &= \pi^{-d}\int_{\mathbf{C}^d} f(z)\overline{g(z)}\, e^{-\|z\|^2}\, dz.\end{aligned}$$

This makes $\mathcal{P}$ into a pre-Hilbert space, and the action

$$f \mapsto f\circ k, \qquad k \in K,$$

is a unitary representation of K on $\mathcal{P}$. It is a deep result of W. Schmidt that this representation has a multiplicity-free decomposition into irreducibles

$$\mathcal{P} = \sum_{\mathbf{m}}^{\oplus} \mathcal{P}_{\mathbf{m}}$$

where $\mathbf{m}$ ranges over all *signatures*, i.e. r-tuples $\mathbf{m} = (m_1, m_2, \dots, m_r) \in \mathbf{Z}^r$ satisfying $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. Polynomials in $\mathcal{P}_{\mathbf{m}}$ are homogeneous of degree $|\mathbf{m}| := m_1 + m_2 + \cdots + m_r$; in particular, $\mathcal{P}_{(0)}$ are the constants and $\mathcal{P}_{(1)}$ the linear polynomials. Any holomorphic function thus has a decomposition $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$, $f_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}$, which refines the usual homogeneous expansion.

Since the spaces $\mathcal{P}_{\mathbf{m}}$ are finite dimensional, they automatically possess a reproducing kernel: there exist functions $K_{\mathbf{m}}(x, y)$ on $\mathbf{C}^d \times \mathbf{C}^d$ such that for each $f \in \mathcal{P}_{\mathbf{m}}$ and $y \in \mathbf{C}^d$, $f(y) = \langle f, K(\cdot, y)\rangle_F$. Explicitly, for any orthonormal basis $\{\psi_j\}_{j=1}^{\dim \mathcal{P}_{\mathbf{m}}}$ of $\mathcal{P}_{\mathbf{m}}$, $K_{\mathbf{m}}$ is given by

$$K_{\mathbf{m}}(x, y) = \sum_{j=1}^{\dim \mathcal{P}_{\mathbf{m}}} \psi_j(x)\overline{\psi_j(y)}. \tag{5.1}$$

It follows from the definition of the $\mathcal{P}_{\mathbf{m}}$ spaces that the kernels $K_{\mathbf{m}}(x, y)$ are K-invariant. By the discussion in the penultimate section, we therefore know that each $K_{\mathbf{m}}$ defines an invariant differential operator

$$\Delta_{\mathbf{m}} f(a) := K_{\mathbf{m}}(\partial, \partial)(f \circ \phi_a)(0), \qquad a \in \Omega. \tag{5.2}$$

Further, one can show that $K_{\mathbf{m}}$ are actually a basis of all K-invariant polynomials, and, consequently, $\Delta_{\mathbf{m}}$ are a linear basis for invariant differential operators. Further, from (5.1) and (5.2) we see that for any f holomorphic,

$$\Delta_{\mathbf{m}} |f|^2(a) = \sum_j |\psi_j(\partial)(f \circ \phi_a)(0)|^2 \geq 0.$$

What makes the basis $\Delta_{\mathbf{m}}$ important for our applications to the Q_ν-spaces is the following converse to the last inequality.

Theorem. *An invariant differential operator*

$$L = \sum_{\mathbf{m}} l_{\mathbf{m}} \Delta_{\mathbf{m}}$$

satisfies $L|f|^2 \geq 0 \ \forall f$ *holomorphic if and only if*

$$l_{\mathbf{m}} \geq 0 \qquad \forall \mathbf{m}.$$

6 Bloch spaces and Q_ν spaces on bounded symmetric domains

Thus we see that the invariant differential operators L that can be used in (4.1) are precisely those which are linear combinations of $\Delta_{\mathbf{m}}$ with nonnegative coefficients. The most basic among such L are evidently the operators $\Delta_{\mathbf{m}}$ themselves. We are thus lead to the following definitions.

Definition. For each signature $\mathbf{m}$, the $\mathbf{m}$-Bloch space is defined by

$$\mathcal{B}_{\mathbf{m}} = \{f \text{ holomorphic on } \Omega: \ \|\Delta_{\mathbf{m}}|f|^2\|_\infty < \infty\}.$$

Definition. For each signature $\mathbf{m}$ and $\nu \in \mathbf{R}$, the space $Q_{\nu,\mathbf{m}}$ is defined by requiring that, for f holomorphic,

$$\begin{aligned} f \in Q_{\nu,\mathbf{m}} &\iff \sup_{\phi\in G} \int_\Omega \Delta_{\mathbf{m}}|f\circ\phi|^2 \, h^\nu \, d\mu < \infty \\ &\iff \sup_{a\in\Omega} \int_\Omega \Delta_{\mathbf{m}}|f\circ\phi_a|^2 \, h^\nu \, d\mu < \infty \\ &\iff \sup_{a\in\Omega} \int_\Omega \Delta_{\mathbf{m}}|f|^2 \, (h\circ\phi_a)^\nu \, d\mu < \infty. \end{aligned}$$

Here $h(z) \equiv h(z,z)$. (All the three conditions above are equivalent owing to the invariance of $\Delta_{\mathbf{m}}$ and $d\mu$.)

Clearly, the above definitions reduce to the usual definitions for $\Omega = \mathbf{D}$ or $\mathbf{B}^d$ and $\mathbf{m} = (1)$ (so that $\Delta_{\mathbf{m}}$ is the invariant Laplacian). In particular, the case of $\mathbf{m} \neq (1)$ gives something new even for the unit disc.

Let us work out the simplest special cases of Bloch spaces.

Example. For $\mathbf{m} = (0)$ we have $\Delta_{\mathbf{m}} = I$, so $\mathcal{B}_{(0)} = H^\infty$. Also,

$$Q_{\nu,(0)} = \begin{cases} H^\infty & \nu > p-1, \\ \{0\} & \nu \le p-1. \end{cases}$$

Example. For $\mathbf{m} = (1)$ we have $\Delta_{(1)} = \widetilde{\Delta}$, so that

$$\mathcal{B}_{(1)} = \{f: \quad \sup_a \|\partial(f\circ\phi_a)(0)\|^2 < \infty\}.$$

This is known as the *Timoney Bloch space* [T].

Example. Assume that Ω is of tube type and $s := d/r \in \mathbf{Z}$. Let $\mathbf{m} = (s, s, \dots, s) \equiv (s^r)$. It is known that in that case the space $\mathcal{P}_{(s^r)} = \mathbf{C}N^s$ is one-dimensional (for the unit disc, $N(z) = z$; for I_{rr}, $N(Z) = \det Z$), the kernel $K_{\mathbf{m}}$ is given (up to a constant factor) by $K_{\mathbf{m}}(\partial,\partial) = N(\partial)^s N(\overline{\partial})^s$, and the so-called Bol's lemma says that for any f holomorphic, $N(\partial)^s(f\circ\phi_a)(0) = \text{const}\cdot h(a)^s N(\partial)^s f(a)$. (For the disc, this reads $(f\circ\phi_a)'(0) = -(1-|a|^2)\, f'(a)$.) Hence,

$$\Delta_{(s^r)}|f|^2 = h^p |N(\partial)^s f|^2 \tag{6.1}$$

and

$$\mathcal{B}_{(s^r)} = \{f \text{ holomorphic: } h^p|N(\partial)^s f|^2 \text{ is bounded}\}.$$

We might call this the *Arazy Bloch space* [A].

7 Example: the polydisc

For clarity, let us also see what is the situation for the bidisc $\Omega = \mathbf{D}^2$. This is definitely NOT a Cartan domain (it is not irreducible), but in many respects it behaves like a Cartan domain with the rank, dimension and genus $r = p = d = 2$, multiplicities $a = b = 0$, and $h(x, y) = (1 - x_1\overline{y}_1)(1 - x_2\overline{y}_2)$. Namely, the invariant measure is $h(z, z)^{-2}\,dz$; the invariant differential operators are precisely the symmetric polynomials in $\widetilde{\Delta}_1, \widetilde{\Delta}_2$, where $\widetilde{\Delta}_j := (1 - |z_j|^2)^2\partial_j\overline{\partial}_j$; the Peter-Weyl spaces are given by $\mathcal{P}_{\mathbf{m}} = \mathbf{C}z_1^{m_1}z_2^{m_2} + \mathbf{C}z_1^{m_2}z_2^{m_1}$, for $\mathbf{m} = (m_1, m_2)$; and, up to a constant factor, $K_{\mathbf{m}}(x, y) = (x_1\overline{y}_1)^{m_1}(x_2\overline{y}_2)^{m_2} + (x_1\overline{y}_1)^{m_2}(x_2\overline{y}_2)^{m_1}$. It follows that

$$\Delta_{(1,0)} = \widetilde{\Delta}_1 + \widetilde{\Delta}_2, \qquad \Delta_{(1,1)} = \widetilde{\Delta}_1\widetilde{\Delta}_2.$$

Thus the Timoney Bloch space $\mathcal{B}_{(1,0)}$ consists of all f holomorphic on $\mathbf{D}^2$ for which

$$(1 - |z_1|^2)^2\Big|\frac{\partial f}{\partial z_1}\Big|^2 + (1 - |z_2|^2)^2\Big|\frac{\partial f}{\partial z_2}\Big|^2 \qquad \text{is bounded,}$$

while the Arazy Bloch space $\mathcal{B}_{(1,1)}$ consists of all f holomorphic on $\mathbf{D}^2$ for which

$$(1 - |z_1|^2)^2(1 - |z_2|^2)^2\Big|\frac{\partial^2 f}{\partial z_1\partial z_2}\Big|^2 \quad \text{is bounded.}$$

We thus see that the Timoney Bloch space is contained in the Arazy Bloch space, and is a proper subset thereof: any holomorphic function of the form $f(z_1, z_2) = g(z_1)$ belongs to $\mathcal{B}_{(1,1)}$, but does not belong to $\mathcal{B}_{(1,0)}$ unless g belongs to the Bloch space on the disc. We also see that the Timoney-Bloch norm vanishes precisely on the constants, while the Arazy-Bloch norm vanishes precisely on functions of the form $f(z_1) + g(z_2)$.

Similar situation can be seen to prevail for a general polydisc $\mathbf{D}^n$: there are n Bloch spaces, of which Timoney is the smallest, and Arazy the largest.

Remark. The big Hankel operator H_f is compact $\iff$ f belongs to the Timoney Bloch space.

8 Composition series

The phenomenon that we have observed for the polydiscs is connected with the existence of the *composition series.* Let us explain this concept on the example of the unit disc $\mathbf{D}$. There the following assertion holds.

Claim. *Let E be any topological vector space of holomorphic functions on $\mathbf{D}$ which is Möbius invariant and on which the group of rotations acts strongly-continuously. Then either $E = \{0\}$, or $E = \{constants\}$, or E contains all polynomials.*

Proof. Let $E \ni f = \sum_{k=0}^{\infty} f_k z^k$; then by rotation invariance,

$$\int_0^{2\pi} f(e^{i\theta}z)\,e^{-mi\theta}\,\frac{d\theta}{2\pi} = f_m z^m \in E. \tag{8.1}$$

Thus if $f_m \neq 0$ for some m, then $z^m \in E$; hence, by invariance, $(\frac{a-z}{1-\overline{a}z})^m \in E\ \forall a \in \mathbf{D}$. Taking this for the f in (8.1) and 0 for the m in (8.1), we get $a^m\mathbf{1} \in E$; thus the constants are in E.

If even $f_m \neq 0$ for some $m \geq 1$, then, applying (8.1) to the same function again but this time taking 1 for the m in (8.1) and noting that $(\frac{a-z}{1-\overline{a}z})^m = a^m - m(1-|a|^2)z + O(z^2)$, we see that $z \in E$; thus by invariance $\frac{a-z}{1-\overline{a}z} = a - (1-|a|^2)z\sum_{j=1}^\infty \overline{a}^j z^j$ belongs to E, for any $a \in \mathbf{D}$. Taking the last function for the f in (8.1) shows that $z^j \in E\ \forall j$, i.e. all polynomials are in E. This completes the proof. $\square$

The last theorem admits the following reformulation. Denote $\mathcal{M}_1 = \{$all holomorphic functions$\}$, $\mathcal{M}_0 = \{$constants$\}$, $\mathcal{M}_{-1} = \{0\}$. Then

$$E \setminus \mathcal{M}_{j-1} \neq \emptyset \implies \mathcal{P} \cap \mathcal{M}_j \subset E.$$

It turns out that, in a sense, precisely the same thing holds for a general Cartan domain.

Namely, let

$$(x)_k := x(x+1)\dots(x+k-1)$$

denote the familiar Pochhammer symbol, and for a signature $\mathbf{m} = (m_1, m_2, \dots, m_r)$, consider the function

$$x \mapsto (x)_{m_1}(x - \frac{a}{2})_{m_2}(x-a)_{m_3}\dots(x - \frac{r-1}{2}a)_{m_r}, \qquad x \in \mathbf{C}.$$

Let $q(\mathbf{m})$ be the multiplicity of zero of this function at $x = 0$:

$$q(\mathbf{m}) := \operatorname{card}\{j :\ m_j > \frac{j-1}{2}a \in \mathbf{Z}\}.$$

Also denote by q the maximum possible value of $q(\mathbf{m})$, i.e.

$$q = \begin{cases} r & a \text{ even}, \\ \left[\dfrac{r+1}{2}\right] & a \text{ odd}. \end{cases}$$

For $-1 \leq j \leq q$, let

$$\mathcal{M}_j = \{f = \sum_{\mathbf{m}} f_{\mathbf{m}} \text{ holomorphic} :\ f_{\mathbf{m}} = 0 \text{ if } q(\mathbf{m}) > j\}.$$

Thus, in particular,

$$\mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \dots \subset \mathcal{M}_q,$$
$$\mathcal{M}_{-1} = \{0\}, \quad \mathcal{M}_0 = \{\text{constants}\}, \quad \mathcal{M}_q = \{\text{all holomorphic}\}.$$

The following deep result is due to Orsted, Faraut and Koranyi.

Theorem. (1) *Each $\mathcal{M}_j$ is G-invariant;*

(2) *for any G-invariant space E of holomorphic functions on which the action of K is strongly continuous,*

$$E \setminus \mathcal{M}_{j-1} \neq \emptyset \implies \mathcal{P} \cap \mathcal{M}_j \subset E.$$

Example. For the bidisc $\mathbf{D}^2$, $q = 2$ and

$$\begin{aligned} &\mathcal{M}_{-1} = \{0\}, \quad \mathcal{M}_0 = \{\text{constants}\}, \\ &\mathcal{M}_1 = f(z_1) + g(z_2), \qquad f, g \text{ holomorphic on } \mathbf{D}, \\ &\mathcal{M}_2 = \{\text{all holomorphic}\}. \end{aligned}$$

9 Results

After all the preparations, we can finally give our results.

Theorem. *If $j < q(\mathbf{m})$, then the $Q_{\nu,\mathbf{m}}$-norm vanishes on $\mathcal{M}_j$; thus $\mathcal{M}_j$ is contained in $Q_{\nu,\mathbf{m}}$ in a trivial way.*

The same is true also for the Bloch space $\mathcal{B}_{\mathbf{m}}$.

Theorem. *If $\nu > p-1$, then $\mathcal{B}_{\mathbf{m}} \subset Q_{\nu,\mathbf{m}}$ continuously.*

Theorem. *If $q(\mathbf{m}) \le q(\mathbf{n})$, then $Q_{\nu,\mathbf{m}} \subset \mathcal{B}_{\mathbf{n}}$ continuously.*

Corollary. $\nu > p-1 \implies Q_{\nu,\mathbf{m}} = \mathcal{B}_{\mathbf{m}}$, *with equivalent norms.*
$q(\mathbf{m}) \le q(\mathbf{n}) \implies \mathcal{B}_{\mathbf{m}} \subset \mathcal{B}_{\mathbf{n}}$ *continuously.*
$q(\mathbf{m}) = q(\mathbf{n}) \implies \mathcal{B}_{\mathbf{m}} = \mathcal{B}_{\mathbf{n}}$, *with equivalent norms.*
$q(\mathbf{m}) = q(\mathbf{n})$, $\nu > p-1 \implies Q_{\nu,\mathbf{m}} = Q_{\nu,\mathbf{m}}$, *with equivalent norms.*

The last corollary exhausts the case $\nu > p-1$ completely. What about $\nu \le p-1$?

Theorem. *If $\nu < 0$, then $Q_{\nu,\mathbf{m}} = \mathcal{M}_{q(\mathbf{m})-1}$.*

Theorem. *For $r > 1$ and $\mathbf{m} = (1,0,\dots,0)$, that is,*

$$f \in Q_{\nu,(1)} \iff \sup_{a\in\Omega} \int_\Omega \widetilde{\Delta} |f\circ\phi_a|^2 \, h^\nu \, d\mu,$$

we have

$$Q_{\nu,(1)} = \begin{cases} \mathcal{B}_{(1)}, \text{ the Timoney Bloch space} & \nu > p-1, \\ \{constants\} & \nu \le p-1. \end{cases}$$

Note that the last theorem means that the situation for $r > 1$ differs radically from the one for $r = 1$ (disc, ball): there Q_ν is nontrivial also for $p-2 < \nu \le p-1$ (for the disc, even for $p-2 \le \nu \le p-1$).

Theorem. *For a tube domain with $s = \frac{d}{r} \in \mathbf{Z}$, and $\mathbf{m} = (s^r)$,*

$$Q_{\nu,(s^r)} = \begin{cases} \mathcal{B}_{(s^r)}, \text{ the Arazy Bloch space} & \nu > p-1 \\ \mathcal{D} & \nu = 0 \\ \mathcal{M}_{q-1} & \nu < 0. \end{cases}$$

Here $\mathcal{D}$ is the Dirichlet space

$$\mathcal{D} = \{f \text{ holomorphic: } N(\partial)^s f \in L^2(\Omega, dz)\}.$$

At the moment, we do not know what are the spaces $Q_{\nu,\mathbf{m}}$ for ν between 0 and $p-1$ and $|\mathbf{m}| > 1$ — for instance, whether they are properly increasing with ν. We can offer somewhat more complete information only for the polydisc:

Theorem. *For the polydisc $\mathbf{D}^r$ (so that $p = 2$, $q(\mathbf{m}) = \#\{j : m_j > 0\}$, and $q = r$),*

$$q(\mathbf{m}) < r \implies Q_{\nu,(\mathbf{m})} = \begin{cases} \mathcal{B}_{\mathbf{m}} & \nu > 1, \\ \mathcal{M}_{q(\mathbf{m})-1} & \nu \le 1; \end{cases}$$

$$q(\mathbf{m}) = r \implies Q_{\nu,(\mathbf{m})} = \begin{cases} \text{Arazy-Bloch} & \nu > 1, \\ \mathcal{D} & \nu = 0, \\ \mathcal{M}_{q(\mathbf{m})-1} & \nu < 0. \end{cases}$$

All this suggests the following conjecture about the nontriviality of the spaces $Q_{\nu,\mathbf{m}}$.

Conjecture. *For tube domains with $s = \frac{d}{r} \in \mathbf{Z}$ and $q(\mathbf{m}) = q$,*

$$Q_{\nu,\mathbf{m}} \iff \nu \geq 0;$$

in all other cases,

$$Q_{\nu,\mathbf{m}} \iff \nu > p-1.$$

10 Some proofs

We proceed to give some hints about the proofs of the theorems.

Theorem. *$j < q(\mathbf{m}) \implies \mathcal{M}_j \subset Q_{\nu,\mathbf{m}}$ and the norm vanishes. Similarly for $\mathcal{B}_{\mathbf{m}}$.*

Proof. Recall that

$$\begin{aligned} f \in Q_{\nu,\mathbf{m}} &\iff \sup_a \int_\Omega \Delta_{\mathbf{m}}|f|^2 \, (h \circ \phi_a)^\nu \, d\mu < \infty, \\ f \in \mathcal{B}_{\mathbf{m}} &\iff \Delta_{\mathbf{m}}|f|^2 \in L^\infty. \end{aligned}$$

Now

$$\begin{aligned} \Delta_{\mathbf{m}}|f|^2(z) &= K_{\mathbf{m}}(\partial,\partial)|f\circ\phi_z|^2(0) \\ &= \sum_j \psi_j(\partial)\overline{\psi_j(\partial)}|f\circ\phi_z|^2(0) \\ &= \sum_j |\psi_j(\partial)(f\circ\phi_z)(0)|^2 \\ &= \sum_j |\langle f\circ\phi_z, \psi_j\rangle_F|^2 \\ &= \|P_{\mathbf{m}}(f\circ\phi_z)\|_F^2, \end{aligned}$$

where $P_{\mathbf{m}}$ denotes the projection onto $\mathcal{P}_{\mathbf{m}}$.

Thus $f \in \mathcal{M}_j \implies f\circ\phi_z \in \mathcal{M}_j \implies P_{\mathbf{m}}(f\circ\phi_z) = 0 \implies \Delta_{\mathbf{m}}|f|^2 = 0 \implies f \in \mathcal{B}_{\mathbf{m}}$ and $f \in Q_{\nu,\mathbf{m}}$. ☐

Theorem. *$\nu > p-1 \implies \mathcal{B}_{\mathbf{m}} \subset Q_{\nu,\mathbf{m}}$ continuously.*

Proof. It is known that for $\nu > p-1$, the measure $h^\nu \, d\mu$ is finite. Thus $\forall a \in \Omega$,

$$\begin{aligned} \int_\Omega (\Delta_{\mathbf{m}}|f|^2)\circ\phi_a \, h^\nu \, d\mu &\leq c_\nu \, \|\Delta_{\mathbf{m}}|f|^2\|_\infty \\ &= c_\nu \|f\|^2_{\mathcal{B}_{\mathbf{m}}}. \end{aligned}$$

☐

Theorem. *$q(\mathbf{m}) \leq q(\mathbf{n}) \implies Q_{\nu,\mathbf{m}} \subset \mathcal{B}_{\mathbf{n}}$ continuously.*

Proof. By the K-invariance of $\Delta_{\mathbf{m}}$ and h, the integral

$$\int_\Omega \Delta_{\mathbf{m}}(f\overline{g})\, h^\nu \, d\mu$$

is a K-invariant bilinear form of $f, g \in \mathcal{P}$. It follows from the Peter-Weyl decomposition of $\mathcal{P}$ into the $\mathcal{P}_{\mathbf{m}}$ that any such bilinear functional must be of the form

$$\sum_{\mathbf{k}} c_{\mathbf{mk}} \, \langle f_{\mathbf{k}}, g_{\mathbf{k}} \rangle_F,$$

for some coefficients $c_{\mathbf{mk}} \geq 0$. Suppose we can show that

$$c_{\mathbf{mn}} > 0. \tag{10.1}$$

Since $\Delta_{\mathbf{n}}|f|^2(0) = \|P_{\mathbf{n}}f\|_F^2 = \|f_{\mathbf{n}}\|_F^2$, it will follow that

$$\Delta_{\mathbf{n}}|f|^2(0) \leq \frac{1}{c_{\mathbf{mn}}} \int_\Omega \Delta_{\mathbf{m}}|f|^2 \, h^\nu \, d\mu.$$

Replacing f by $f \circ \phi_a$, this becomes

$$\Delta_{\mathbf{n}}|f|^2(a) \leq \frac{1}{c_{\mathbf{mn}}} \int_\Omega \Delta_{\mathbf{m}}|f \circ \phi_a|^2 \, h^\nu \, d\mu.$$

Taking suprema over all $a \in \Omega$ gives the assertion.

It remains to prove (10.1). But by the properties of the composition series,

$$\begin{aligned}
c_{\mathbf{mn}} = 0 \iff & \int_\Omega \Delta_{\mathbf{m}}|f_{\mathbf{n}}|^2 \, h^\nu \, d\mu = 0 \qquad \forall f_{\mathbf{n}} \in \mathcal{P}_{\mathbf{n}} \\
\iff & \Delta_{\mathbf{m}}|f_{\mathbf{n}}|^2(z) = 0 \quad \forall z \, \forall f_{\mathbf{n}} \\
\iff & \|P_{\mathbf{m}}(f_{\mathbf{n}} \circ \phi_z)\|_F^2 = 0 \quad \forall z \, \forall f_{\mathbf{n}} \\
\iff & P_{\mathbf{m}}(f_{\mathbf{n}} \circ \phi_z) = 0 \quad \forall z \, \forall f_{\mathbf{n}} \\
\iff & P_{\mathbf{m}}\mathcal{M}_{q(\mathbf{n})} = 0 \\
\iff & q(\mathbf{m}) > q(\mathbf{n}).
\end{aligned}$$

□

Theorem. $\nu < 0 \implies Q_{\nu,\mathbf{m}} = \mathcal{M}_{q(\mathbf{m})-1}$.

Proof. From the composition series we know that

$$\begin{aligned}
\mathcal{M}_{q(\mathbf{m})-1} \subsetneqq Q_{\nu,\mathbf{m}} &\implies \mathcal{P} \cap \mathcal{M}_{q(\mathbf{m})} \subset Q_{\nu,\mathbf{m}} \\
&\implies \mathcal{P}_{\mathbf{m}} \subset Q_{\nu,\mathbf{m}} \\
&\implies \sup_a \int_\Omega \Delta_{\mathbf{m}}|f|^2 \, (h \circ \phi_a)^\nu \, d\mu < \infty \qquad \forall f \in \mathcal{P}_{\mathbf{m}}.
\end{aligned}$$

Since $K_{\mathbf{m}}(z,z) = \sum_j |\psi_j(z)|^2$ for any basis $\{\psi_j\}$ of $\mathcal{P}_{\mathbf{m}}$, we can continue by

$$\implies \sup_a \int_\Omega \Delta_{\mathbf{m}} K_{\mathbf{m}} \cdot (h \circ \phi_a)^\nu \, d\mu < \infty$$

where $K_{\mathbf{m}} = K_{\mathbf{m}}(z,z)$. It can be shown that

$$\exists m \gg 0: \quad \Delta_{\mathbf{m}} K_{\mathbf{m}} \geq c\, h^m.$$

Thus we can continue by

$$\Longrightarrow \sup_a \int_\Omega h^m\, (h\circ\phi_a)^\nu\, d\mu < \infty. \tag{10.2}$$

Forelli-Rudin inequalities show that this happens iff $\nu \geq 0$. □

Theorem. *For rank $r>1$ and $\mathbf{m}=(1,0,\dots,0)$,*

$$Q_{\nu,(1)} = \begin{cases} \mathcal{B}_{(1)}, \ \textit{the Timoney Bloch space} & \nu > p-1, \\ \{\text{constants}\} & \nu \leq p-1. \end{cases}$$

Proof. As above,

$$\{\text{constants}\} \subsetneq Q_{\nu,(1)} \iff \sup_a \int_\Omega \widetilde{\Delta}\|z\|^2\, (h\circ\phi_a)^\nu\, d\mu < \infty.$$

The fact that

$$\widetilde{\Delta}\|z\|^2 \approx \begin{cases} h^2 & \Omega = \mathbf{D}, \\ h & \Omega = \mathbf{B}^d, \\ 1 & r>1 \end{cases}$$

and (10.2) again yield the conclusion. □

Theorem. *For a tube domain with $s=\frac{d}{r}\in\mathbf{Z}$, and $\mathbf{m}=(s^r)$,*

$$Q_{\nu,(s^r)} = \begin{cases} \mathcal{B}_{(s^r)}, \ \textit{the Arazy Bloch space} & \nu > p-1 \\ \mathcal{D} & \nu = 0 \\ \mathcal{M}_{q-1} & \nu < 0. \end{cases}$$

Proof. As mentioned before, in this case

$$\Delta_{\mathbf{m}} = h^p N(\partial)^s N(\overline{\partial})^s$$

for a certain polynomial N (the Jordan norm). Hence

$$\begin{aligned} f \in Q_{\nu,\mathbf{m}} &\iff \sup_a \int_\Omega h^p\, |N(\partial)^s f|^2\, (h\circ\phi_a)^\nu\, d\mu < \infty \\ &\iff \sup_a \int_\Omega |N(\partial)^s f|^2\, (h\circ\phi_a)^\nu\, dz < \infty. \end{aligned}$$

Thus for $\nu \geq 0$, all polynomials belong to $Q_{\nu,\mathbf{m}}$.

For $\nu = 0$, this coincides with the definition of the Dirichlet space.

The case $\nu < 0$ was settled by the previous theorem. □

Theorem. *For the polydisc* $\mathbf{D}^r$ *(so that* $p=2$*,* $q(\mathbf{m})=\#\{j : m_j>0\}$*, and* $q=r$*),*

$$q(\mathbf{m})<r \implies Q_{\nu,(\mathbf{m})}=\begin{cases}\mathcal{B}_{\mathbf{m}} & \nu>1,\\ \mathcal{M}_{q(\mathbf{m})-1} & \nu\le 1;\end{cases}$$

$$q(\mathbf{m})=r \implies Q_{\nu,(\mathbf{m})}=\begin{cases}\textit{Arazy-Bloch} & \nu>1,\\ \mathcal{D} & \nu=0,\\ \mathcal{M}_{q(\mathbf{m})-1} & \nu<0.\end{cases}$$

Proof. Using explicit formulas for $K_{\mathbf{m}}$, $\Delta_{\mathbf{m}}$ etc. given in one of the preceding sections, this is easily reduced to explicit calculations on the disc. □

11 Open problems

We conclude the paper by a list of open problems.

(1) The first of them is, of course, to determine when $Q_{\nu,\mathbf{m}}$ is nontrivial — we repeat here the conjecture stated above:

Conjecture. $Q_{\nu,\mathbf{m}}$ *nontrivial iff*
$\nu\ge 0$ *(for tube domain with* $\frac{d}{r}\in\mathbf{Z}$ *and* $q(\mathbf{m})=q$*)*
$\nu>p-1$ *(otherwise).*

(2) If $q(\mathbf{m})=q(\mathbf{n})$, is $Q_{\nu,\mathbf{m}}=Q_{\nu,\mathbf{n}}$? (We have seen that this holds for the Bloch spaces, hence also for $\nu>p-1$; the case of $\nu\le p-1$ remains unresolved.)

(3) If $\nu_1<\nu_2$ and $Q_{\nu_1,\mathbf{m}}, Q_{\nu_2,\mathbf{m}}$ are nontrivial, is $Q_{\nu_1,\mathbf{m}}\subsetneq Q_{\nu_2,\mathbf{m}}$? (For $\mathbf{D}$, this was proved in [AXZ], and for $\mathbf{B}^d$ in [AC].)

(4) In principle, one can define $Q_{\nu,L}$ and $\mathcal{B}_L$ for any invariant differential operator L, even when the right-hand side in (3) is not nonnegative, by

$$f\in\mathcal{B}_L \iff L|f|^2 \text{ is bounded},$$

$$f\in Q_{\nu,L} \iff \sup_a\int_\Omega \left|L|f\circ\phi_a|^2\right| h^\nu\,d\mu<\infty.$$

If L is such that $L|f|^2\ge 0$ for all f holomorphic f, i.e. if

$$L=\sum_{\mathbf{m}} l_{\mathbf{m}}\,\Delta_{\mathbf{m}},\qquad l_{\mathbf{m}}\ge 0, \tag{11.1}$$

then it is easy to see that

$$Q_{\nu,L}=\bigcap_{\mathbf{m}:\,l_{\mathbf{m}}>0} Q_{\nu,\mathbf{m}},$$

$$\mathcal{B}_L=\mathcal{B}_{\mathbf{m}}\qquad \text{where } q(\mathbf{m})=\min\{q(\mathbf{k}):l_{\mathbf{k}}>0\}.$$

What happens for operators L not satisfying (11.1)?

For instance, does the space of all holomorphic f on $\mathbf{D}$ for which

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}\left|\widetilde{\Delta}^2|f(\phi_a(z)|^2\right|(1-|z|^2)^{\nu-2}\,dz<\infty$$

coincide with the Bloch space for $\nu>1$?

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Commutative C^*-algebras of Toeplitz Operators on the Unit Disk

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We give a complete characterization of commutative C^*-algebras of Toeplitz operators acting on weighted Bergman spaces over the unit disk. This note is a short version of the paper [7], where all proofs and details can be found.

1 Commutative algebras and hyperbolic geometry

We introduce the following Möbius invariant normalized measure on the unit disk $\mathbb{D}$

$$d\mu(z) = \frac{1}{\pi}\frac{dx \wedge dy}{(1-(x^2+y^2))^2} = \frac{1}{2\pi i}\frac{d\overline{z}\wedge dz}{(1-|z|^2)^2}.$$

For $h \in (0,1)$, the weighted Bergman space $\mathcal{A}^2_h(\mathbb{D})$ on the unit disk (see, for example, [3]) is the space of analytic functions in $L_2(\mathbb{D}, d\mu_h)$, where

$$d\mu_h(z) = (\frac{1}{h}-1)(1-|z|^2)^{\frac{1}{h}}\, d\mu(z),$$

and

$$\|f\|_h = \left(\int_{\mathbb{D}} |f(z)|^2\, d\mu_h(z)\right)^{\frac{1}{2}}.$$

Note, that for $h = \frac{1}{2}$ we have the classical weightless Bergman space $\mathcal{A}^2(\mathbb{D})$ (with normalized measure).

The orthogonal Bergman projection from $L_2(\mathbb{D}, d\mu_h)$ onto the weighted Bergman space has the form (see, for example, [3]):

$$(B^{(h)}_{\mathbb{D}} f)(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\overline{\zeta})^{\frac{1}{h}}}\, d\mu_h(\zeta) = (\frac{1}{h}-1)\int_{\mathbb{D}} f(\zeta)\left(\frac{1-\zeta\overline{\zeta}}{1-z\overline{\zeta}}\right)^{\frac{1}{h}} d\mu(\zeta).$$

Given a function $a(z) \in L_\infty(\mathbb{D})$, the Toeplitz operator $T^{(h)}_a$ with symbol a is defined on $\mathcal{A}^2_h(\mathbb{D})$ as follows

$$T^{(h)}_a : \varphi \in \mathcal{A}^2_h(\mathbb{D}) \longmapsto B^{(h)}_{\mathbb{D}}(a\varphi) \in \mathcal{A}^2_h(\mathbb{D}).$$

It was recently shown ([11, 12]) that apart from the known case of radial symbols in the classical (weightless) Bergman space $\mathcal{A}^2(\mathbb{D})$ there exists a rich family of *commutative* C^*-algebras of Toeplitz operators. Moreover, surprisingly it turns out that these commutative properties of Toeplitz operators do not depend at all on smoothness properties of the symbols: the corresponding symbols can be merely measurable. Furthermore it turns out ([4, 5, 6]) that the above classes of symbols generate commutative C^*-algebras of Toeplitz operators on *each weighted Bergman space* $\mathcal{A}_h^2(\mathbb{D})$. The prime cause here appears to be the geometric configuration of level lines of symbols.

In this context it is useful to consider the unit disk $\mathbb{D}$ as the hyperbolic plane equipped with the standard hyperbolic metric

$$ds^2 = \frac{1}{\pi} \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}.$$

Recall that a geodesic, or a hyperbolic straight line, on $\mathbb{D}$ is a part of an Euclidean circle or of a straight line orthogonal to the boundary of $\mathbb{D}$.

Each pair of geodesics, say L_1 and L_2, determines (see, for example, [1]) a geometrically defined object, a one-parameter family $\mathcal{P}$ of geodesics, which is called the *pencil* defined by L_1 and L_2. Each pencil has an associated family $\mathcal{C}$ of lines, called *cycles*, which are the orthogonal trajectories to geodesics forming the pencil.

The pencil $\mathcal{P}$ defined by L_1 and L_2 is called

1. *parabolic* if L_1 and L_2 are parallel (and tend to the same point $z_0 \in \partial\mathbb{D}$), in this case $\mathcal{P}$ is the set of all geodesics parallel to L_1 and L_2, and the cycles are called *horocycles*;

2. *elliptic* if L_1 and L_2 are intersecting (at a point $z_0 \in \mathbb{D}$), in this case $\mathcal{P}$ is the set of all geodesics passing through the common point of L_1 and L_2;

3. *hyperbolic* if L_1 and L_2 are disjoint, in this case $\mathcal{P}$ is the set of all geodesics orthogonal to the unique common orthogonal geodesic (with endpoints z_1, $z_2 \in \partial\mathbb{D}$) of L_1 and L_2, and the cycles are called *hypercycles*.

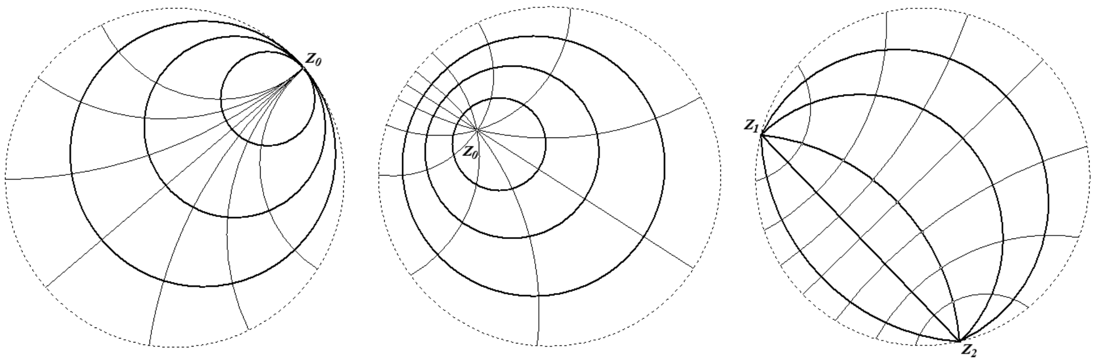

FIGURE 1. Parabolic, elliptic and hyperbolic pencils.

In Figure 1, illustrating possible pencils, the cycles are drawn in bold lines.

The following main theorem has been proved in [11, 12] for the classical (weightless) Bergman space, and in [4, 5, 6] for all weighted Bergman spaces.

Theorem 1.1. *Given a pencil $\mathcal{P}$ of geodesics, consider the set of L_∞-symbols which are constant on corresponding cycles. The C^*-algebra generated by Toeplitz operators with such symbols is commutative on each weighted Bergman space $\mathcal{A}_h^2(\mathbb{D})$.*

2 Three-term asymptotic expansion formula

To get an inverse statement to Theorem 1.1 we will use the familiar Berezin quantization procedure on the unit disk (see, for example, [2, 3]).

For each function $a = a(z) \in C^\infty(\mathbb{D})$ consider the family of Toeplitz operators $T_a^{(h)}$ with (anti-Wick) symbol a acting on $\mathcal{A}_h^2(\mathbb{D})$, for $h \in (0,1)$. The Wick symbols of the Toeplitz operator $T_a^{(h)}$ has the form

$$\widetilde{a}_h(z,\overline{z}) = (\frac{1}{h}-1)\int_{\mathbb{D}} a(\zeta)\left(\frac{(1-|z|^2)(1-|\zeta|^2)}{(1-z\overline{\zeta})(1-\zeta\overline{z})}\right)^{\frac{1}{h}} d\mu(\zeta),$$

and the star product of Wick symbols is defined as follows

$$(\widetilde{a}_h \star \widetilde{b}_h)(z,\overline{z}) = (\frac{1}{h}-1)\int_{\mathbb{D}} \widetilde{a}_h(z,\overline{\zeta})\,\widetilde{b}_h(\zeta,\overline{z})\left(\frac{(1-|z|^2)(1-|\zeta|^2)}{(1-z\overline{\zeta})(1-\zeta\overline{z})}\right)^{\frac{1}{h}} d\mu(\zeta).$$

To achieve our goal we need the three-term asymptotic expansion formula of the commutator of two Wick symbols.

Theorem 2.1. *For any pair $a = a(z,\overline{z})$ and $b = b(z,\overline{z})$ of six times continuously differentiable functions the following three-term asymptotic expansion formula holds*

$$\begin{aligned}\widetilde{a}_h \star \widetilde{b}_h - \widetilde{b}_h \star \widetilde{a}_h \;\; &= \;\; i\hbar\,\{a,b\} + i\frac{\hbar^2}{4}\left(\Delta\{a,b\} + \{a,\Delta b\} + \{\Delta a, b\} + 8\pi\{a,b\}\right)\\ &\quad + i\frac{\hbar^3}{24}\big[\{\Delta a,\Delta b\} + \{a,\Delta^2 b\} + \{\Delta^2 a, b\} + \Delta^2\{a,b\}\\ &\quad + \Delta\{a,\Delta b\} + \Delta\{\Delta a, b\} + 28\pi\left(\Delta\{a,b\} + \{a,\Delta b\} + \{\Delta a,b\}\right)\\ &\quad + 96\pi^2\{a,b\}\big] + o(\hbar^3),\end{aligned}$$

where $\hbar = \frac{h}{2\pi}$, and the Poisson bracket and the Laplace-Beltrami operator are given by

$$\begin{aligned}\{a,b\} &= 2\pi i(1-z\overline{z})^2\left(\frac{\partial a}{\partial z}\frac{\partial b}{\partial \overline{z}} - \frac{\partial a}{\partial \overline{z}}\frac{\partial b}{\partial z}\right),\\ \Delta &= 4\pi(1-z\overline{z})^2\frac{\partial^2}{\partial z\partial\overline{z}}.\end{aligned}$$

Corollary 2.2. *Let $\mathcal{A}(\mathbb{D})$ be a subalgebra of $C^\infty(\mathbb{D})$ such that for each $h \in (0,1)$ the Toeplitz operator algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ is commutative. Then for all a, $b \in \mathcal{A}(\mathbb{D})$ we have*

$$\{a,b\} = 0, \tag{2.1}$$

$$\{a,\Delta b\} + \{\Delta a, b\} = 0, \tag{2.2}$$

$$\{\Delta a,\Delta b\} + \{a,\Delta^2 b\} + \{\Delta^2 a, b\} = 0. \tag{2.3}$$

3 Consequences of (2.1), (2.2), and (2.3)

Discussing commutative C^*-algebras of Toeplitz operators we will always assume that the corresponding generating class of symbols is a linear space. To underline the geometric nature of symbol classes which generate the commutative C^*-algebras of Toeplitz operators we have considered bounded measurable symbols in Theorem 1.1. This also agrees with the desire for such (commutative) algebras to be, in a sense, maximal. Note that the arguments used in the proof do not require any assumption on smoothness properties of symbols. The same result (commutativity of Toeplitz operator C^*-algebra) remains valid for any linear subspace of L_∞-symbols (constant on cycles). Moreover, we can start with a much more restricted set of symbols (say, smooth symbols only) and extend them furthermore to all L_∞-symbols by means of uniform and strong operator limits of sequences of Toeplitz operators.

As we consider the C^*-algebra generated by Toeplitz operators, we can always assume, without loss of generality, that our set of symbols is closed under complex conjugation and contains the function $e(z) \equiv 1$.

Let $\mathcal{A}(\mathbb{D})$ be a linear space of (smooth) functions. Denote by $\mathcal{T}(\mathcal{A}(\mathbb{D})) = \{\mathcal{T}_h(\mathcal{A}(\mathbb{D}))\}_h$ the family of C^*-algebras $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ generated by Toeplitz operators with symbols from $\mathcal{A}(\mathbb{D})$ and acting on the weighted Bergman spaces $\mathcal{A}^2_h(\mathbb{D})$.

To introduce our symbol classes we need the notion of the jet of a function (see, for example, [9, 10]). Given two complex valued smooth functions f and g defined in a neighborhood of a point $z \in \mathbb{D}$, we say that they have the same jet of order k at z if their real partial derivatives at z up to order k are equal. It is easy to see that such relation does not depend on the coordinate system and that it defines an equivalence relation. The corresponding equivalence class of a function f at z is denoted by $j^k_z(f)$ and is called the k-th order jet of f at z. Furthermore, given a complex vector space $\mathcal{A}(\mathbb{D})$ of smooth functions, we denote with $J^k_z(\mathcal{A}(\mathbb{D}))$ the space of k-jets at z of the elements in $\mathcal{A}(\mathbb{D})$. We observe that $J^k_z(\mathcal{A}(\mathbb{D}))$ is a finite dimensional complex vector space.

In what follows, for a differentiable function $f : \mathbb{D} \to \mathbb{C}$ we will say that $z \in \mathbb{D}$ is a nonsingular point of f if $df_z \neq 0$.

The symbol classes that we are considering are given in the next definition.

Definition 3.1. *Let $\mathcal{A}(\mathbb{D})$ be a complex vector space of smooth functions. We will say that $\mathcal{A}(\mathbb{D})$ is k-rich if it is closed under complex conjugation and the following conditions are satisfied:*

(i) there is a finite set S such that for every $z \in \mathbb{D} \setminus S$ at least one element of $\mathcal{A}(\mathbb{D})$ is nonsingular at z,

(ii) for every point $z \in \mathbb{D} \setminus S$ and $l = 0, \ldots, k$, the space of jets $J^l_z(\mathcal{A}(\mathbb{D}))$ has complex dimension at least $l + 1$.

As the set $\mathcal{A}(\mathbb{D})$ is closed under the complex conjugation, it is sufficient to consider the conditions (2.1), (2.2), and (2.3) for real valued functions only. Recall that each real valued function $a \in \mathcal{A}(\mathbb{D})$, nonsingular in some open set, has in this set two systems of mutually orthogonal smooth lines, the system of level lines and the system of gradient lines.

The geometric information contained in the first term of asymptotic expansion of a commutator, or equivalently in the condition (2.1), is given by the next lemma.

Lemma 3.2. *Let $\mathcal{A}(\mathbb{D})$ be a 2-rich space of smooth functions which generates for each $h \in (0, 1)$ the commutative C^*-algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ of Toeplitz operators. Then all real valued functions in $\mathcal{A}(\mathbb{D})$ have (globally) the same set of level lines and the same set of gradient lines.*

Vanishing of the second term of asymptotic in a commutator, or equivalently the condition (2.2), leads to the following theorem.

Theorem 3.3. *Let $\mathcal{A}(\mathbb{D})$ be a 2-rich space of smooth functions which generates for each $h \in (0,1)$ the commutative C^*-algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ of Toeplitz operators. Then the common gradient lines of all real valued functions in $\mathcal{A}(\mathbb{D})$ are geodesics in the hyperbolic geometry of the unit disk $\mathbb{D}$.*

Vanishing of the third term of asymptotic in a commutator, or equivalently the condition (2.3), implies the following theorem.

Theorem 3.4. *Let $\mathcal{A}(\mathbb{D})$ be a 3-rich vector space of smooth functions $\mathcal{A}(\mathbb{D})$ which generates for each $h \in (0,1)$ the commutative C^*-algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ of Toeplitz operators. Then the common level lines of all real valued functions in $\mathcal{A}(\mathbb{D})$ are cycles.*

The next theorem provides a geometric characterization of the real valued functions on $\mathbb{D}$ whose gradient lines define a pencil of geodesics.

Theorem 3.5. *A nonconstant C^3 real valued function a in $\mathbb{D}$ defines a pencil if and only if the following two conditions are satisfied:*

(i) The gradient lines of a are geodesics.

(ii) Each level line of a is a cycle.

Lemma 3.2, Theorem, 3.3, Corollary 3.4, and Theorem 3.5 lead directly to the following result.

Corollary 3.6. *Let $\mathcal{A}(\mathbb{D})$ be a 3-rich vector space of smooth functions such that $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ is commutative for each $h \in (0,1)$. Then there exists a pencil $\mathcal{P}$ of geodesics in $\mathbb{D}$ such that all functions in $\mathcal{A}(\mathbb{D})$ are constant on the cycles of $\mathcal{P}$.*

Now the main result of the paper reads as follows.

Corollary 3.7. *Let $\mathcal{A}(\mathbb{D})$ be a 3-rich vector space of smooth functions. Then the following three statements are equivalent:*

(i) there is a pencil $\mathcal{P}$ of geodesics in $\mathbb{D}$ such that all functions in $\mathcal{A}(\mathbb{D})$ are constant on the cycles of $\mathcal{P}$;

(ii) the C^*-algebra generated by Toeplitz operators with $\mathcal{A}(\mathbb{D})$-symbols is commutative on each weighted Bergman space $\mathcal{A}_h^2(\mathbb{D})$, $h \in (0,1)$.

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Berezin–Toeplitz quantization of the moduli space of flat $SU(n)$ connections

Martin Schlichenmaier

1 Introduction

This is a condensed write-up of a talk presented at the program "Complex Analysis, Operator Theory, and Applications to Mathematical Physics" organized by F. Haslinger, E. Straube, and H. Upmeier at the Erwin-Schrödinger-Institute in Vienna in September 2005, and at the "Conference on Poisson Geometry", organized by T. Ratiu, A. Weinstein, and N.T. Zung at the ICTP, Trieste, in 2005.

First, we recall the basics of the Berezin-Toeplitz quantization (operator and formal deformation quantization). Then we discuss the moduli space of flat $SU(n)$ connections on a fixed Riemann surface in its different guises. Finally, we present recent results obtained by Jørgen Andersen showing the asymptotic faithfulness of the representations of the mapping class group (MCG, Teichmüller group) on the covariantly constant sections of the projectivized Verlinde bundle. In his approach he uses the Toeplitz operators and results on their correct semiclassical behavior as they will be presented in the first part.

As far as the Berezin-Toeplitz quantization is concerned the results are results obtained partly in joint works with E. Meinrenken, and M. Bordemann resp. with A. Karabegov [3], [9], [10], [11], [8].

Quite a number of mathematician (and physicists) were involved in the study of the moduli space of connections and the mapping class group. Instead giving references here, let me refer to the recent overviews by L. Jeffrey [5] and G. Masbaum [6]. The beautiful results on the asymptotic faithfulness presented are entirely due to Andersen [1]. For similar results for the $U(1)$ obtained by him see also [2].

2 BT quantization of compact Kähler manifolds

2.1 Kähler manifolds

Let (M, ω) be a Kähler manifold, i.e. M a complex manifold, and ω a a closed $(1,1)$-form on M which is positive (a Kähler form).

Examples:

1. $\mathbb{C}^n$, $\qquad \omega = \mathrm{i}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$,
2. $\mathbb{P}^1$, $\qquad \omega = \frac{\mathrm{i}}{(1+z\bar{z})^2} dz \wedge d\bar{z}$,
3. every Riemann surface carries a Kähler form,
4. every (complex) torus of arbitrary dimension with the standard Kähler form on $\mathbb{C}^n$ (see 1.),

5. every (quasi-)projective manifold, i.e. every non-empty open subset of a projective variety without singularities, with the restriction of the Fubini-Study Kähler form of the projective space,

6. very often moduli spaces in the algebraic or analytic context carry a natural Kähler structure coming from their construction.

2.2 Quantizable Kähler manifolds

Definition 2.1. *(Quantization condition) A Kähler manifold (M,ω) is called quantizable, if there exists an associated quantum line bundle (L,h,∇), i.e. a holomorphic line bundle L over M, with hermitian metric h on L, and compatible connection ∇, fulfilling*

$$curv_{L,\nabla} = -\mathrm{i}\,\omega.$$

Note: Not all Kähler manifolds are quantizable. For example only such tori are quantizable which have enough theta functions, i.e. which can be embedded holomorphically into projective space. They are called abelian varieties.

For the rest of this write-up we assume that M is a compact Kähler manifold. We fix a quantum line bundle L and consider $L^m := L^{\otimes m}$, with metric $h^{(m)}$, and take

$$\Gamma_\infty(M,L^m) \quad \text{the space of smooth global sections, and}$$

$$\Gamma_{hol}(M,L^m) = \mathrm{H}^0(M,L^m) \quad \text{the subspace of global holomorphic sections.}$$

Due to the compactness of M, the latter is finite-dimensional. On these spaces a scalar product is defined via

$$\langle\varphi,\psi\rangle := \int_M h^{(m)}(\varphi,\psi)\,\Omega, \qquad \Omega := \frac{1}{n!}\underbrace{\omega\wedge\omega\cdots\wedge\omega}_{n}.$$

We will need the projector

$$\Pi^{(m)} : L^2(M,L^m) \longrightarrow \Gamma_{hol}(M,L^m).$$

2.3 Berezin-Toeplitz operator quantization

Fix $f\in C^\infty(M)$, and let $s\in\Gamma_{hol}(M,L^m)$ then the following map

$$s \quad\mapsto\quad T_f^{(m)}(s) := \Pi^{(m)}(f\cdot s)$$

defines the *Toeplitz operator of level m*

$$T_f^{(m)} : \quad \Gamma_{hol}(M,L^m) \to \Gamma_{hol}(M,L^m).$$

The *Berezin-Toeplitz (BT) operator quantization* is the map

$$f \mapsto \left(T_f^{(m)}\right)_{m\in\mathbb{N}_0}.$$

The reason to call it a quantization is, that it has the correct semi-classical behavior as expressed in the following theorem.

Theorem 2.1. *(Bordemann, Meinrenken, and Schlichenmaier (BMS) [3])*
(a)

$$\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty, \tag{2.1}$$

(b)

$$||m\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| = O(1/m), \tag{2.2}$$

(c)

$$||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| = O(1/m). \tag{2.3}$$

The proofs of (b) and (c) are based on the symbol calculus of generalized Toeplitz operators developed by Boutet de Monvel and Guillemin [4].

2.4 Deformation quantization

Theorem 2.1. *(BMS, Schl., Karabegov and Schl.) [3], [9], [10], [11], [8]. There exists a unique differential star product, the BT star product,*

$$f \star_{BT} g = \sum \nu^k C_k(f,g), \tag{2.4}$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T_{C_k(f,g)}^{(m)}, \quad (m \to \infty). \tag{2.5}$$

This star product is of "separation of variables" type, and has classifying Deligne-Fedosov class

$$\frac{1}{\mathrm{i}}\left(\frac{1}{\lambda}[\omega] - \frac{\epsilon}{2}\right), \tag{2.6}$$

and Karabegov form

$$\frac{-1}{\lambda}\omega + \omega_{can}, \tag{2.7}$$

A star product is a differential star product if the $C_k(.,.)$ are bidifferential operators in their function arguments. Such a differential star product is of "separation of variables type" if the first argument is only differentiated in holomorphic directions and the second argument only in anti-holomorphic directions (resp. the opposite directions depending on the convention chosen). This notion is due to Karabegov [7], and corresponds to the fact that the star product respects the complex structure. Such star products are classified by their formal Karabegov form. Above λ is used as formal variable for the forms and the formal forms are formal power series in λ if we ignore $1/\lambda$ which comes with the fixed ω. In particular, for the BT star product no higher formal powers of λ occur. The Deligne-Fedosov class is a formal H^2_{deRahm} class which classifies the star product up to equivalence. The form ω_{can} is the curvature form of the canonical (holomorphic) line bundle with fibre metric coming from the Liouville form.

Note also that the asymptotic formula (2.5) is a short-hand notation for a very precise and strong asymptotic behaviour of the norms of the involved operators. See the cited references for the precise statement.

3 The moduli space of flat $SU(n)$ connections

3.1 Its symplectic structure

Let X be an oriented compact surface, and $p \in X$ a fixed point. We denote by G the group $SU(n)$.

Let $\mathcal{A}_{F,\xi}$ be the set of flat $SU(n)$ connections over $X \setminus \{p\}$ with holonomy ξ of finite order d around p. We fix for the center of $SU(n)$ a generator and identify it with $\mathbb{Z}/n\mathbb{Z}$. Then ξ corresponds to $d \bmod n$.

The group of maps $X \to G$ from the surface X to the group G with pointwise multiplication in G, is the gauge group $\mathcal{G}$. It acts on the connection via gauge transformations

$$A^g := g^{-1}dg + g^{-1}Ag. \tag{3.1}$$

The moduli space of connections is the quotient of the set of connections modulo these gauge transformations

$$\mathcal{M} := \mathcal{A}_{F,\xi}/\mathcal{G} \cong Hom_d(\tilde{\pi}_1(X), G)/G \tag{3.2}$$

The latter equivalence is the fact that this moduli space can be identified with the space of those group homomorphisms of the central extension $\tilde{\pi}_1(X)$ of the fundamental group $\pi_1(X)$ defined by

$$0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{\pi}_1(X) \longrightarrow \pi_1(X) \longrightarrow 0$$

with values in G, for which the generator $1 \in \mathbb{Z}$ in the central extension is mapped to $d \bmod n$ in the center of G, where the homomorphism are identified modulo conjugation in G.

Let $\mathcal{M}_s$ be the moduli space of irreducible flat connections (this corresponds to irreducible representations). It is a manifold, carries a natural symplectic structure ω, and an associated hermitian line bundle $\mathcal{L}$ which is a quantum line bundle with respect to the symplectic structure. It is constructed from the WZW cocycle of the Chern-Simons action. See the appendix for more details and [5] for references and further information.

3.2 Its complex structure

We choose a complex structure σ on X. This structure will induce complex structures on all introduced objects.

1. $X \Longrightarrow X^\sigma$ is now a (compact) Riemann surface,
2. $(\mathcal{M}_s, \omega) \Longrightarrow (\mathcal{M}_s^\sigma, \omega^\sigma)$ is now a Kähler manifold,
3. $\mathcal{L} \Longrightarrow \mathcal{L}^\sigma$ becomes a hermitian holomorphic line bundle, in fact, it is a quantum line bundle with respect to ω^σ.

Hence, $\mathcal{M}_s^\sigma$ is a quantizable Kähler manifold with quantum line bundle $\mathcal{L}^\sigma$. But what is the geometry of $\mathcal{M}_s^\sigma$? Is it compact? To study these questions we discuss another description of the moduli space.

4 Holomorphic rank n bundles E over smooth projective curves

4.1 The moduli space

Recall that the compact Riemann surface X^σ can be identified with a smooth projective curve C over $\mathbb{C}$. In the following we consider holomorphic vector bundles over C. First we define for every rank n holomorphic vector bundle E, its determinant line bundle as $\det E := \bigwedge^n E$, and its degree as $\deg(E) := \deg(\det E)$. The question is: *Does there exist a moduli space of isomorphy classes of such bundles?* The answer is: *In generally not!* We need to restrict our considerations to the subset of isomorphy classes of *(Mumford) stable* bundles, resp. S-equivalence classes of *semi-stable* bundles. A bundle E is stable (resp. semi-stable) iff for every non-trivial subbundle F of E one has $\deg(F)/rk(F) < \deg(E)/rk(E)$ (resp. $\le$). For the S-equivalence relation two semi-stable (but not stable) bundles are identified if certain associated graded objects are isomorphic.

Let T be a line bundle and $n \in \mathbb{N}$. We use the following notation for the moduli space of bundles

$U_s(n,d)$, rk $(E) = n$, $\deg(E) = d$, E stable,
$U_s(n,T)$, rk $(E) = n$, $\det(E) = T$, E stable,
$U(n,d)$, rk $(E) = n$, $\deg(E) = d$, E semi-stable,
$U(n,T)$, rk $(E) = n$, $\det(E) = T$, E semi-stable.

In the following let $[p]$ be the line bundle corresponding to the divisor p, i.e. the line bundle which has a non-trivial section with exactly a zero of order one at p and which is non-vanishing elsewhere. Furthermore let $d[p]$ be its d-tensor power. In particular, $\deg d[p] = d$.

We have the following properties:

1. $M := U(n, d[p])$ is always projective algebraic (hence compact),
2. $M_s := U_s(n, d[p])$ is Zariski open and smooth in M, hence a smooth manifold,
3. if $gcd(n,d) = 1$ then $M = M_s$, and hence M_s is a compact Kähler manifold,
4. the singularities of M are rather mild,
5. for the Picard group of isomorphy classes of line bundles we have $Pic(M_s) = Pic(M) = \mathbb{Z}\cdot[L]$, where L is a special ample line bundle,
6. $\Gamma_{hol}(M_s, L^m_{|}) = \Gamma_{hol}(M, L^m)$,
7. if $g = 2$ and $n = 2$ then M is always smooth.

The fundamental result is
$$\boxed{\mathcal{M}_s^\sigma \cong U_s(n, d[p]) = M_s}$$
as complex manifold and as Kähler manifolds, and
$$\boxed{\mathcal{L}^\sigma \cong L}$$
as holomorphic line bundles.
A few names of people involved are Narasimhan, Seshadri, Weil, Mumford, ...

4.2 The Verlinde bundle

The Verlinde spaces are the vector spaces $\mathrm{H}^0(M, L^m) = \Gamma_{hol}(M, L^m)$ and the dimension formula (as function of m) is called the Verlinde formula.

These Verlinde spaces are the quantum spaces, and the BT operators

$$T_f^{(m)} : \mathrm{H}^0(M, L^m) \to \mathrm{H}^0(M, L^m)$$

are the quantum operators. We can apply Theorem 2.1 (BMS) und use the natural deformation quantization $\star_{BT}$ of Theorem 2.1 (at least without modification, if $M = M_s$, resp. if M is smooth).

We have to go one step further. If we consider the following diagram we see that the first line does not depend on the complex structure σ, but the second does.

$$\begin{array}{ccccc} X & \longrightarrow & (\mathcal{M}_s, \mathcal{L}^m) & \longrightarrow & \Gamma_\infty(\mathcal{M}_s, \mathcal{L}^m) \\ \Big\downarrow \text{choose } \sigma & & \Big\downarrow & & \Big\downarrow \Pi^{\sigma,(m)} \\ X^\sigma & \longrightarrow & (\mathcal{M}_s^\sigma = M_s, (\mathcal{L}^\sigma)^m = L^m) & \longrightarrow & \Gamma_{hol}(M_s, L^m) \end{array}$$

If we vary our σ over the Teichmüller space $\mathcal{T}$, (i.e. the space of all complex structures on X modulo a certain equivalence relation) the first line will give trivial families of objects, the second line nontrivial families over $\mathcal{T}$.

In particular, over $\mathcal{T}$ there is the trivial (infinite dimensional) bundle with fibre $\Gamma_\infty(\mathcal{M}_s, \mathcal{L}^m)$ which contains the subbundle $\mathcal{V}_m$ with fibre $\Gamma_{hol}(M_s, L^m)$. The bundle $\mathcal{V}_m$ is the called the *Verlinde bundle* over $\mathcal{T}$

Given $f \in C^\infty(\mathcal{M}_s)$ its Toeplitz operator depends on the complex structure. Hence,

$$\left(T_{f,\sigma}^{(m)}\right)_{\sigma \in \mathcal{T}}$$

is a family of operators on the Verlinde bundle. In other words $T_{f,\,.}^{(m)}$ is a section of $End(\mathcal{V}_m)$.

5 The mapping class group (MCG) action

Over Teichmüller space $\mathcal{T}$ we have the bundles $\mathcal{V}_m$ and $End(\mathcal{V}_m)$. We will discuss the following points:

1. There exists a naturally defined projectively flat connection ∇ on $\mathcal{V}_m$, it is the Axelrod, della Pietra, Witten – Hitchin connection.
2. The MCG operates on the covariantly constant sections of $\mathbb{P}(\mathcal{V}_m)$.
3. J. Andersen showed that this action of the MCG is asymptotically faithful (i.e. given an element of the MCG, there is an m such that the element operates non-trivially).

Recall that the mapping class group(MCG) is defined as

$$\Gamma := MCG := Diff^+(X)/Diff_0(X),$$

here X is the surface of genus g, $Diff^+(X)$ the group of orientation preserving diffeomorphisms and $Diff_0(X)$ the subgroup of diffeomorphisms which are isotop to the identity.

1. By definition Γ operates on the surface X.
2. It operates on the Teichmüller space. In fact the moduli space $\mathcal{M}_g$ of isomorphism classes of compact genus g Riemann surfaces (resp. smooth projective curves of genus g) is the quotient $\mathcal{T}/\Gamma$.
3. It operates on the fundamental group $\pi_1(X)$, and on $Hom_d(\tilde{\pi}_1(X), G)/G$.
4. And furthermore it operates on $\mathcal{M}_s^\sigma \cong M_s$, the moduli spaces of irreducible connections, resp. stable bundles.

5.1 Andersen's result

Let ∇ be the Axelrod-della Pietra-Witten – Hitchin (AdPW-H) connection on $\mathcal{V}_m$ which is projectively flat. It induces a flat connection ∇^{end} on $End(\mathcal{V}_m)$. We denote by $\mathbb{P}(W_m)$, the space of covariantly constant sections of $\mathbb{P}(\mathcal{V}_m)$ with respect to ∇. Then the MCG operates also on $\mathbb{P}(W_m)$:

$$\rho_m : \Gamma \to Aut(\mathbb{P}(W_m)).$$

Theorem 5.1. *(Andersen, [1]) For $g \geq 3$ the map ρ_m is asymptotically faithful. More precisely,*

$$\bigcap_{m=1}^{\infty} ker(\rho_m) = \begin{cases} 1, & g > 2, \ or\ g = 2, n > 2, \ or \\ & \qquad g = 2, n = 2, d \ odd, \\ \{1, H\}, & g = 2, n = 2, d \ even, \end{cases} \tag{5.1}$$

where H is the hyperelliptic involution.

5.2 Importance

The assignment

$$X \quad \longrightarrow \quad V(X) = \mathrm{H}^0(M_s, L^m)$$

corresponds to a Topological Quantum Field Theory (TQFT). It should be independent of the complex structure chosen. The projectively flat connection gives locally a natural identification. Globally the choice reduces to on action of the mapping class group Γ — (which is also a topological invariant). Hence, this action gives invariants of the TQFT in question.

5.3 The relation to BT

Note in the following that $f \in C^\infty(\mathcal{M}_s)$, i.e. f is a smooth function on the moduli space of connections, resp. bundles.

Proposition 5.1. *(Andersen, [1]) For $\sigma_0, \sigma_1 \in \mathcal{T}$, denote by $P^{end}_{\sigma_0,\sigma_1}$ the parallel transport from σ_0 to σ_1 in $End(\mathcal{V}_m)$, then*

$$||P^{end}_{\sigma_0,\sigma_1} T^{(m)}_{f,\sigma_0} - T^{(m)}_{f,\sigma_1}|| = O(1/m). \tag{5.2}$$

He uses Theorem 2.1 (BMS), the deformation quantization of Theorem 2.1 and carries out further ingenious hard work.

Proposition 5.2. *(Andersen, [1]) Let $\phi \in \Gamma$, such that $\phi \in ker \rho_m$, then*

$$T^{(m)}_{f,\sigma} = P^{end}_{\phi(\sigma),\sigma} T^{(m)}_{f\circ\phi,\phi(\sigma)} \tag{5.3}$$

Theorem 5.3. *(Andersen, [1]) Let $\phi \in \Gamma$, such that $\phi \in \bigcap_{m\in\mathbb{N}} \ker \rho_m$, then ϕ induces the identity on $\mathcal{M}_s$.*

Proof. By Proposition 5.2 and the linearity in the function argument of the Toeplitz operators we have

$$T^{(m)}_{f-f\circ\phi,\sigma} = T^{(m)}_{f,\sigma} - T^{(m)}_{f\circ\phi,\sigma} = P^{end}_{\phi(\sigma),\sigma} T^{(m)}_{f\circ\phi,\phi(\sigma)} - T^{(m)}_{f\circ\phi,\sigma}.$$

We take the norm of this expression and use Proposition 5.1:

$$||T^{(m)}_{f-f\circ\phi,\sigma}|| = O(1/m).$$

Or,

$$\lim_{m\to\infty} ||T^{(m)}_{f-f\circ\phi,\sigma}|| = 0.$$

This implies $|f - f\circ\phi|_\infty = 0$ by Theorem 2.1, Part a, for all f, hence $\phi = id$ considered as element acting on the moduli space. □

Theorem 5.1 follows from known results which elements of the mapping class group act trivially on the moduli space of connections.

6 Appendix: Symplectic form on $\mathcal{M}$

Let $\mathcal{A}_F$ be the affine space of all flat $SU(n)$ connections, and $\mathfrak{g} = su(n)$. The tangent vectors at $A \in \mathcal{A}_F$ can be given as $\alpha, \beta \in \Omega^1(X) \otimes \mathfrak{g}$. On this space

$$\Omega_A(\alpha,\beta) = \frac{\mathrm{i}}{2\pi}\int_X \mathrm{Tr}(\alpha\wedge\beta)$$

is a skew-symmetric form which is invariant under the gauge group and hence descends to $\mathcal{M} = \mathcal{A}_F/\mathcal{G}$. If we restrict the situation to the irreducible connections, then the quotient $\mathcal{M}_s = \mathcal{A}^s_F/\mathcal{G}$ is a manifold and Ω descends to a symplectic form on $\mathcal{M}_s$

To define the bundle one uses the Chern-Simons(CS) action. Let N be a 3-manifold with boundary $\partial N = X$. For any connection $\tilde{A}$ on N

$$CS(\tilde{A}) := \frac{1}{4\pi}\int_N \mathrm{Tr}(\tilde{A}\wedge d\tilde{A} + \frac{2}{3}\tilde{A}\wedge\tilde{A}\wedge\tilde{A}).$$

For a connection on X we take any extension $\tilde{A}$ to N. Also for a gauge transformation $g \in \mathcal{G}$ we take any extension $\tilde{g}: N \to G$. Then

$$\theta(A,g) := \exp(\mathrm{i}(CS(\tilde{A}^{\tilde{g}}) - CS(\tilde{A}))$$

is a $U(1)$-valued well-defined cocycle (the WZW cocycle). It is used to construct the bundle $\mathcal{L}$ over $\mathcal{M}_s$ as quotient

$$\mathcal{L} := (\mathcal{A}^s_F \times \mathbb{C})/\sim \quad\to\quad \mathcal{A}^s_F/\mathcal{G} = \mathcal{M}_s$$

where $(A,z) \sim (A^g, \theta(A,g)z)$. The one form $\eta(\alpha) = \frac{1}{4\pi}\int_X \mathrm{Tr}(A\wedge\alpha)$ on $\mathcal{A}_F$ induces a unitary connection on $\mathcal{L}$, whose curvature is essentially equal to the symplectic form.

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Kontsevich Quantization and Duflo Isomorphism

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The aim of this expository note is to explain the relationship between the Kontsevich's Formality theorem and the problem of local solvability of bi-invariant differential operators on finite-dimensional real Lie groups. Main references to this subject are [5, 1, 2, 3, 6, 7].

Let G be a connected real finite dimensional Lie group and $\mathfrak{g}$ be its Lie algebra. The symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ can be seen as the algebra of differential operators with constant coefficients on $\mathfrak{g}$ as well as the algebra (with respect to the convolution on the vector space $\mathfrak{g}$) of distributions on $\mathfrak{g}$ supported at the origin, it is always commutative. On the other hand side the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ can be seen as the algebra of left-invariant differential operators on G as well as the algebra (with respect to the convolution on G) of distributions on G supported at the identity element. The algebra $U(\mathfrak{g})$ is neither commutative (except when $\mathfrak{g}$ is commutative) nor graded. However $U(\mathfrak{g})$ is filtered by the order of differential operators and the Poincaré-Birkhoff-Witt theorem ensures that its associated graded algebra $grU(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$. This isomorphism, called *symmetrization*, is given by $\beta(X_1...X_n) = \frac{1}{n!}\sum_{\sigma\in S_n} X_{\sigma(1)}...X_{\sigma(n)}$. Obviously $S(\mathfrak{g})$ and $U(\mathfrak{g})$ are not isomorphic as algebras so one cannot "transform" the convolution of distributions on the Lie algebra into the convolution of distributions on the corresponding Lie group and thus to reduce the solvability of left-invariant differential operators on G to the solvability of differential operators with constant coefficients on $\mathfrak{g}$. However, it turns out that the set of $\mathrm{ad}(\mathfrak{g})$-invariants in $S(\mathfrak{g})$ is isomorphic as an algebra to the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. This remarkable fact was described for reductive Lie algebras by Harish-Chandra and for nilpotent ones by Dixmier. The validity of this fact for arbitrary real finite-dimensional Lie algebras was established by Duflo [4] in 1979. A highly non trivial proof of this result was based on the orbit method that relates coadjoint orbits of G (parametrized by invariant symmetric tensors) with irreducible representations of G (whose infinitesimal characters are elements of $Z(\mathfrak{g})$). More precisely, let

$$q(x) = \det_{\mathfrak{g}}\left(\frac{\sinh(\frac{\mathrm{ad}x}{2})}{\frac{\mathrm{ad}x}{2}}\right)^{\frac{1}{2}}, \tag{0.1}$$

be a formal power series on $\mathfrak{g}^*$ and ∂_q be the differential operator of infinite order on $\mathfrak{g}$ with symbol q. Then, the so-called Duflo map $\beta\circ\partial_q : S(\mathfrak{g}) \to U(\mathfrak{g})$ is a vector space isomorphism that becomes algebra isomorphism when restricted to invariants. Therefore the convolution of invariant distributions on G can be recovered form the convolution of their invariant pull-backs on $\mathfrak{g}$ and thus every non-zero bi-invariant differential operator on G admits a local fundamental solution.

We shall explain how does this theorem follow from the formality theorem and how can it be extended in cohomology.

1 Formality theorem and its consequences

Let X be a smooth manifold. One associates to X two graded differential Lie algebras (GDLA). The first GDLA $\mathfrak{g}_1 = T_{\mathrm{poly}}(X)$ is the graded algebra of poly-vector fields on X: $T^n_{\mathrm{poly}}(X) := \Gamma(X, \Lambda^{n+1}TX)$, $\quad n \geq -1$, equipped with the Schouten-Nijenhuis bracket $[\ ,\ ]_{SN}$ and the differential $d := 0$.

Recall that the Schouten-Nijenhuis bracket is given for all $k, l \geq 0$, $\xi_i, \eta_j \in \Gamma(X, TX)$ by:

$$[\xi_0 \wedge \cdots \wedge \xi_k, \eta_0 \wedge \cdots \wedge \eta_l]_{SN} = \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{i+j} [\xi_i, \eta_j] \wedge$$
$$\xi_0 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \ldots \wedge \xi_k \wedge \eta_0 \wedge \cdots \wedge \eta_{j-1} \wedge \eta_{j+1} \wedge \cdots \wedge \eta_l$$

And for $k \geq 0$ and $h \in \Gamma(X, \mathcal{O}_X)$, $\xi_i \in \Gamma(X, TX)$:

$$[\xi_0 \wedge \cdots \wedge \xi_k, h]_{SN} = \sum_{i=0}^{k} (-1)^i \xi_i(h) \cdot (\xi_0 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \ldots \wedge \xi_k)\,.$$

Where $[\xi_i, \eta_j]$ is the usal vector fields bracket *i.e.* the Lie derivative $L_{\xi_i}(\eta_j)$.

The second GDLA associated to X is the algebra of poly-differential operators $\mathfrak{g}_2 = D_{\mathrm{poly}}(X)$ seen as a sub-algebra of the shifted Hochschild complex of functions algebra of X. The grading on $D_{\mathrm{poly}}(X)$ is given by $|A| = m - 1$ where $A \in D_{\mathrm{poly}}(X)$ is a $m-$differential operator. The composition of two operators $A_1 \in D^{m_1}_{\mathrm{poly}}(X)$ and $A_2 \in D^{m_2}_{\mathrm{poly}}(X)$ is given for $f_i \in \mathcal{O}_X$ by:

$$\begin{aligned}(A_1 \circ A_2)(f_1, \ldots, f_{m_1+m_2-1}) \quad &= \quad \sum_{j=1}^{m_1} (-1)^{(m_2-1)(j-1)} A_1(f_1, \ldots, f_{j-1}, \\ &\qquad A_2(f_j, \ldots, f_{j+m_2-1}), f_{j+m_2}, \ldots, f_{m_1+m_2-1}).\end{aligned}$$

One defines the Gerstenhaber bracket: $[A_1, A_2]_G := A_1 \circ A_2 - (-1)^{|A_1||A_2|} A_2 \circ A_1$. Thus the differential on $D_{\mathrm{poly}}(X)$ is given by $dA = -[\mu, A]_G$, where μ is the bi-differential operator of multiplication: $\mu(f_1, f_2) = f_1 f_2$.

Certainly these GDLA are not isomorphic, however the map $\mathcal{U}_1^{(0)} : T_{\mathrm{poly}} \mapsto D_{\mathrm{poly}}$ given by

$$\mathcal{U}_1^{(0)} : (\xi_0 \wedge \cdots \wedge \xi_n) \mapsto \left(f_0 \otimes \cdots \otimes f_n \to \sum_{\sigma \in S_{n+1}} \frac{\mathrm{sgn}(\sigma)}{(n+1)!} \prod_{i=0}^{n} \xi_{\sigma(i)}(f_i) \right) \tag{1.1}$$

for $n \geq 0$ and by $f \mapsto (1 \to f)$ for $f \in \Gamma(X, \mathcal{O}_X)$ is a quasi-isomorphism of complexes. This statement is a version of the Kostant-Hochschild-Rosenberg theorem. It turns out, see [5], that one can extend this map to an application that is a GDLA-morphism up to homotopy.

Consider shifted algebras $\mathfrak{g}_1[1]$ and $\mathfrak{g}_2[1]$ and the coalgebras without unit

$$S^+\big(\mathfrak{g}_i[1]\big) = \bigoplus_{n \geq 0} S^n\big(\mathfrak{g}_i[1]\big), \quad i = 1, 2$$

Both of them have co-derivations Q^i of degree 1 defined by the GDLA structure.

Theorem 1. *There exists a L_∞-quasi-isomorphism between shifted GDLA $\mathfrak{g}_1[1]$ and $\mathfrak{g}_2[1]$, i.e. a co-algebra morphism*

$$\mathcal{U} : S^+\big(\mathfrak{g}_1[1]\big) \to S^+\big(\mathfrak{g}_2[1]\big)$$

such that

$$\mathcal{U} \circ Q^1 = Q^2 \circ \mathcal{U}$$

and such that the restriction $\mathcal{U}$ to $\mathfrak{g}_1[1] \simeq S^1(\mathfrak{g}_1[1])$ is the quasi-isomorphism of co-chain complex $\mathcal{U}_1^{(0)}$ given by (1.1).

This construction is based on an explicit realization of the L_∞-quasi-isomorphism $\mathcal{U}$ in the flat case. We shall now recall it, see [5].

Let X be the vector space $\mathbb{R}^d$. Then one shows that such a L_∞-quasi-isomorphism $\mathcal{U}$ is determined by its "Taylor coefficients" $\mathcal{U}_k : S^k(\mathfrak{g}_1[1]) \to \mathfrak{g}_2[1]$, with $k \geq 1$ that one gets composing $\mathcal{U}$ with the canonical projection $\pi : S^+(\mathfrak{g}_2) \to \mathfrak{g}_2$. One denotes $\overline{\mathcal{U}}$ this composition.

Coefficients $\mathcal{U}_n$ are described in terms of graphs and their weights.

Let $G_{n,m}$ be the set of labeled oriented graphs with n verices of the first kind and m vertices of the second kind such that:

1. All edges start from vertices of the first kind.
2. There are no loops.
3. There are no double edges.

One says that such a graph is *admissible*. By labeling an admissible graph Γ one understands a total order on the set E_Γ of edges of Γ compatible with the order on the set of vertices.

Consider $\Gamma \in G_{n,m}$ and denote by s_k the number of edges starting from the vertex of the first kind with label k. To any n-uplet $(\alpha_1, \dots, \alpha_n)$ of poly-vector fields on X such that for all $k = 1, \dots, n$ the element α_k is a s_k-vector field, one associates, a m-differential operator $B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n)$, given by the following construction : let $\{e_k^1, \dots, e_k^{s_k}\}$ be an ordered sub-set of E_Γ of edges starting from the vertex of the first kind k. To every map $I : E_\Gamma \to \{1, \dots, d\}$ and every vertex x of Γ one associates the differential operator with constant coefficients : $D_{I(x)} = \prod_{e=(-,x)} \partial_{I(e)}$, where for all $i \in \{1, \dots, d\}$ one denotes by ∂_i the partial derivative with respect to the i-th coordinate. The product is taken for all edges ending at x. Let α_k^I be the coefficient :

$$\alpha_k^I = \alpha_k^{I(e_k^1)\cdots I(e_k^{s_k})} = \langle \alpha_k, \, dx_{I(e_k^1)} \wedge \cdots \wedge dx_{I(e_k^{s_k})} \rangle = \langle \alpha_k, dx_{I(e_k^1)} \otimes \cdots \otimes dx_{I(e_k^{s_k})} \rangle.$$

One set :

$$B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n)(f_1 \otimes \cdots \otimes f_m) = \sum_{I : E_\Gamma \to \{1, \dots, d\}} \prod_{k=1}^{n} D_{I(k)} \alpha_k^I \prod_{l=1}^{m} D_{I(\bar{l})} f_l.$$

The Taylor coefficient $\mathcal{U}_n$ is given then by :

$$\mathcal{U}_n(\alpha_1, \dots, \alpha_n) = \sum_{\Gamma \in G_{n,m}} w_\Gamma B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n), \tag{1.2}$$

where the sum is taken over all admissible graphs Γ such that the corresponding operator $B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n)$ is well defined and the integer m is defined by $m-2 = \sum_{k=1}^{n} s_k - 2n$. The coefficient w_Γ is a certain weight associated to every graph Γ. Then $\mathcal{U}_n(\alpha_1, \ldots, \alpha_n)$ is a $m-$differential operator.

The weight w_Γ is zero unless the number of edges $|E_\Gamma|$ of Γ is equal to $2n+m-2$. It is given by integration of a closed form Ω_Γ of degree $|E_\Gamma|$ on a connected component of the Fulton-McPherson compactification of a configuration space whose dimension is precisely $2n+m-2$, see [5]. This weight does depend on the order chosen on the set of edges, but the product $w_\Gamma \cdot B_\Gamma$ does not.

More precisely one denotes $\mathrm{Conf}_{n,m}$ the set of $(p_1, \ldots, p_n, q_1, \ldots, q_m)$ where the p_j are distinct points in the Poincaré upper half-plane : $\mathbb{H}_+ = \{z \in \mathbb{C}, \mathrm{Im}\, z > 0\}$, and q_j are distinct points in $\mathbb{R}$ seen as the boundary of $\mathbb{H}_+$. The group : $G = \{z \mapsto az + b \text{ with } (a,b) \in \mathbb{R} \text{ and } a > 0\}$ acts freely on $\mathrm{Conf}_{n,m}$. The coset : $C_{n,m} = \mathrm{Conf}_{n,m}/G$ is a $2n+m-2$-dimensional manifold. Kontsevich described compactifications $\overline{C_{n,m}}$ of these configuration manifolds. They are $2n+m-2$-dimensional manifolds with corners such that the boundary components in $\overline{C}_{n,m} \setminus C_{n,m}$ correspond to various degenerations of the configurations of points.

Consider, for example, the space

$$C_{2,0} = \{(p_1, p_2) \in \mathbb{H}_+^2 \,|p_1 \neq p_2\}/G^1.$$

For each point $c \in C_{2,0}$ we can choose a unique representative of the form $(\sqrt{-1}, z) \in \mathrm{Conf}_{2,0}$. Thus $C_{2,0}$ is homeomorphic to $\mathbb{H}_+ \setminus \{\sqrt{-1}\}$.

Similarly, it is easy to see that

$$C_2 \simeq S^1, \quad C_{1,1} \simeq (0,1), \quad C_{0,2} \simeq \{0,1\}.$$

and

$$\overline{C}_2 = C_2, \quad \overline{C}_{1,1} = C_{1,1} \sqcup C_{0,2} = [0,1].$$

Of particular interest is the space $\overline{C}_{2,0} = C_{2,0} \sqcup (C_{0,2} \sqcup C_{1,1} \sqcup C_{1,1} \sqcup C_2)$, which can be drawn as “the Eye”.

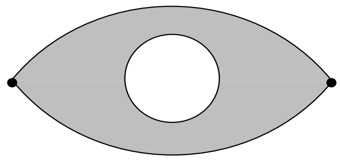

Figure 1: $\overline{C}_{2,0}$ (the Eye)

The circle C_2 represents two points coming close together in the interior of $\mathbb{H}_+$, and two arcs $C_{1,1} \sqcup C_{1,1}$ represent the first point (or the second point) coming close to the real line. Finally, the two corners $C_{0,2}$ correspond to both points approaching the real line.

For every graph $\Gamma \in G_{n,m}$ one defines an angular function : $\Phi_\Gamma : \overline{C}_{n,m} \longrightarrow (\mathbb{R}/2\pi\mathbb{Z})^{|E_\Gamma|}$ in the following way : one draws the graph in $\overline{\mathbb{H}_+}$ joining vertices by geodesics with respect to the hyperbolic metric, and to every edge $e = (p,q)$ one associates the angle $\varphi_e = Arg\left(\frac{q-p}{q-\bar{p}}\right)$ formed by the vertical line passing through p and the edge e (see [2]).

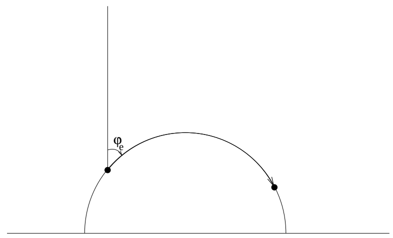

Figure 2: Angular function

Choosing an order on edges this defines Φ_Γ on $C_{n,m}$ and one checks that this map can be extended to the compactification. Let Ω_Γ be the differential form $\Phi_\Gamma^*(dv)$ on $\overline{C}_{n,m}$ where dv is the normalized volume form on $(\mathbb{R}/2\pi\mathbb{Z})^{|E_\Gamma|}$. Let $\overline{C}^+_{n,m}$ be the connected component of $\overline{C}_{n,m}$. The set $\overline{C}^+_{n,m}$has natural orientation and one defines the weight w_Γ by :

$$w_\Gamma = \int_{\overline{C}^+_{n,m}} \Omega_\Gamma. \tag{1.3}$$

1.1 Tangent quasi-isomorphism and homotopy

Let $\gamma \in T_{\mathrm{poly}}(\mathbb{R}^d)[1]$ be a 2−vector field such that $\gamma[-1]$ satisfies the Maurer-Cartan equation in $T_{\mathrm{poly}}(\mathbb{R}^d)$. Thus it is a Poisson 2−vector field. The Kontsevich's L_∞-quasi-isomorphism starting with $\hbar\gamma$ gives rise to a star-product $\star_{\hbar\gamma}$:

$$\star_{\hbar\gamma} = \mu + \overline{\mathcal{U}}(\hbar\gamma) = \mu + \sum_{n\geq 1} \frac{\hbar^n}{n!}\mathcal{U}_n(\gamma, \dots, \gamma), \tag{1.4}$$

where $\hbar$ is a formal parameter. Indeed, a star-product is associative if, as an element of $\mathfrak{g}_2$, it satisfies a Maurer-Cartan type equation, so $\mathcal{U}$ relies two solutions of Maurer-Cartan equations in corresponding GDLA's. Notice that the associativity results from the Stokes theorem applied to the integrals defining the weights w_Γ.

Example. Let γ be the constant Poisson 2-vector field given by a constant symplectic structure Λ on $\mathbb{R}^{2n}$. Then the only non-vanishing graphs are those whose vertices of the first kind do not receive incoming edges. Thus the corresponding star-product is precisely the Moyal star-product:

$$f \star_M g\,(z) = \exp(\hbar\,\Lambda^{rs}\partial_{x_r}\partial_{x_s})(f(x)g(y))_{|x=y=z}.$$

Identifying GDLA's $\mathfrak{g}_1$ and $\mathfrak{g}_2$ with their tangent spaces one defines the derivate map $d\overline{\mathcal{U}} : T_{\mathrm{poly}}(\mathbb{R}^d)[1] \to D_{\mathrm{poly}}(\mathbb{R}^d)[1][[\hbar]]$ at the point $\hbar\gamma$:

$$d\overline{\mathcal{U}}_{\hbar\gamma}(\delta) := \sum_{n>0} \frac{\hbar^{n-1}}{(n-1)!}\mathcal{U}_n(\delta.\gamma^{\cdot n}). \tag{1.5}$$

Tangent spaces to GDLA's $\mathfrak{g}_1$ and $\mathfrak{g}_2$ inherit differential structures in such a way that the co-boundary operator of the tangent co-chain complex of $T_{\mathrm{poly}}(X)[1]$ is given by

$Q^{\hbar\gamma} = -[\hbar\gamma, -]_{SN}$. this is a graded derivative of the exterior product $\wedge$ of poly-vector fields. This exterior product induces an associative and commutative product $\cup$ (called cup-product) on the cohomology space $H_{\hbar\gamma}$ of the first tangent space.

On the second tangent space one introduces an associative graded product :

$$(A_1 \cup A_2)(f_1 \otimes \cdots \otimes f_{m_1+m_2}) = \\ A_1(f_1 \otimes \cdots \otimes f_{m_1}) \star_{\hbar\gamma} A_2(f_{m_1+1} \otimes \cdots \otimes f_{m_2}) \quad (1.6)$$

for every m_1-differential operator A_1 and m_2-differential operator A_2. This operation is compatible with the co-boundary $[-, \star]_G$ of the second tangent complex and it induces a cup-product on the cohomology space $H_{\overline{\mathcal{U}}(\hbar\gamma)}$.

The following theorem is due to Kontsevich and was carefully proven by Manchon and Torossian, see [5] §8 and [6] Théorème 1.2.

Theorem 2. *Let $X = \mathbb{R}^d$ and $\mathcal{U}$ be the Kontsevich's L_∞-quasi-isomorphism. The differential $d\overline{\mathcal{U}}_{\hbar\gamma}$ induces an algebra isomorphism from the cohomology space $H_{\hbar\gamma}$ of the tangent space $T_{\hbar\gamma}(\mathfrak{g}_1[1])$ onto the cohomology space $H_{\overline{\mathcal{U}}(\hbar\gamma)}$ of the tangent space $T_{\overline{\mathcal{U}}(\hbar\gamma)}(\mathfrak{g}_2[1])$.*

I.e. for every pair (α, β) of poly-vector fields such that $[\alpha, \gamma]_{SN} = [\beta, \gamma]_{SN} = 0$ one has

$$d\mathcal{U}^0(\alpha \cup \beta) = d\mathcal{U}^0(\alpha) \cup d\mathcal{U}^0(\beta) + D, \quad (1.7)$$

where D is the Hochschild co-boundary of the algebra $(C^\infty(X)[[\hbar]], \star_\gamma)$ given by $D = -[\star_\gamma, d\mathcal{U}^1(\alpha, \beta)]_G$.

2 Quantization of the Kirillov-Kostant-Poisson bracket

In the case when the manifold X is the dual of a finite dimensional Lie algebra $\mathfrak{g}$ coefficients of the canonical Kirillov-Kostant-Poisson 2-vector field γ are linear functions on $\mathfrak{g}^*$. Let $\{e_1, \ldots, e_d\}$ be a basis of $\mathfrak{g}$ and $(e_1^*, \ldots, e_d^*)$ be its dual basis, then the associated Poisson 2-vector field is given by:

$$\gamma = \frac{1}{2} \sum_{i,j} [e_i, e_j] e_i^* \wedge e_j^*.$$

Therefore on can considerably simplify the expression of poly-differential operators B_Γ that occur in the definition of the star-product $\star_\gamma$ (1.4). Because of the linearity of coefficients of γ the only remaining graphs are those whose vertices of the first kind receive at most one incoming edge.

Let f_1 and f_2 be two polynomials on $\mathfrak{g}^*$ with $\deg(f_1) = l_1$, $\deg(f_2) = l_2$. Then we remark that the graphs contributing to the star-product formula for $f_1 \star f_2$ can have no more than $l_1 + l_2$ vertices of the first type. Indeed, for any $\Gamma \in A_n$ the corresponding bi-differential operator $B_{\Gamma,\gamma}$ contains exactly $2n$ differentiations. When $2n > n + l_1 + l_2$, B_Γ is obviously 0 (because f_1 can be differentiated at most l_1 times, f_2 at most l_2 times and each of the coefficients of γ corresponding to the remaining vertices at most once). Hence

$$f_1 \star f_2 = \sum_{n=0}^{l_1+l_2} \frac{\hbar^n}{n!} \sum_{\Gamma \in A_n} w_\Gamma B_\Gamma(f_1, f_2).$$

This sum is finite, and if we set $\hbar = 1$, we obtain a polynomial on $\mathfrak{g}^*$ of degree l_1+l_2. Therefore, Kontsevich's $\star$-product descends to an *actual product* on the algebra of polynomial functions on $\mathfrak{g}^*$, which can be naturally identified with the symmetric algebra $\mathcal{S}(\mathfrak{g})$.

In [5] §8.4, Kontsevich denotes by I_{alg} the algebra isomorphism between $(S(\mathfrak{g}), \star_\gamma)$ and $U(\mathfrak{g})$and shows that the identification of $(S(\mathfrak{g}), \star_\gamma)$ with $U(\mathfrak{g})$ is given precisely by the Duflo isomorphism of vector spaces.

Notice now that the zero tangent cohomology of $\mathfrak{g}_1$ which is in this case the zero Poisson cohomology of the symmetric algebra $S(\mathfrak{g})$ is precisely the set of $\mathrm{ad}(\mathfrak{g})$-invariants in $S(\mathfrak{g})$ and on the other hand side that the zero tangent cohomology of $\mathfrak{g}_2$ is the zero Hochshild cohomology of $\mathfrak{g}$ with coefficients in $U(\mathfrak{g})$ that is nothing else but the center $Z(\mathfrak{g})$ of the universal enveloping algebra. Therefore, according to the theorem 2 the differential $d\overline{\mathcal{U}}_{\hbar\gamma}$ of the Kontsevich's L_∞-quasi-isomorphism gives rise to an algebra isomorphism between these to set which is precisely the Duflo map (0.1).

The fact that the Duflo map extends to all tangent cohomology groups was shown in [7].

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Quantization Restrictions for Diffeomorphism Invariant Gauge Theories

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Abstract

The classical Stone-von Neumann theorem guarantees the uniqueness of the kinematical framework of quantum mechanics. In this short article we review the corresponding situation for diffeomorphism invariant field theories like gravity. Within the loop quantization framework, we first describe the Weyl algebra and sketch then a fundamental uniqueness result for its representations.

1 Introduction

The quantization of a classical system can be performed in different ways. Nevertheless, there are typical features common to many of them. Often, the major step consists of three choices: First, one selects some set $\mathcal{V}$ of classical variables to be "quantized". Second, one picks out some Hilbert space $\mathfrak{H}$. Finally, one chooses some assignment between the selected classical variables and some operators on $\mathfrak{H}$. It is a fundamental question – both mathematically and physically –, how far these three choices are unique or may be made freely. Unfortunately, in general, this question cannot be answered adequately. First of all, the selection of $\mathcal{V}$ is obviously highly non-unique. At the same time, it very much constrains the two latter steps. In fact, if $\mathcal{V}$ is too small, we may loose, e.g., information about some physical degrees of freedom; if, however, $\mathcal{V}$ is too large, van Hove arguments may show that there is no (nontrivial) quantization at all. Therefore, we will restrict ourselves to the two latter steps, i.e., the "representation theory" of some given $\mathcal{V}$.

The focal point of the present paper is to study the situation in gauge field theories that incorporate diffeomorphism invariance. This is of enormous relevance, in particular, for the quantization of gravity. There are several approaches to this issue. We will consider here the loop quantization. This mainly means to include nonlocal parallel transports along paths (or loops) instead of the connections themselves into the set $\mathcal{V}$ of classical variables to be quantized. We are going to review under which assumptions the kinematical framework used there is indeed unique. Before, however, we will motivate our considerations by the celebrated Stone-von Neumann theorem in quantum mechanics. Proven some 75 years ago, it is responsible for the (to a large extent) uniqueness of quantization of classical mechanics.

In classical mechanics in, for brevity, one dimension, the configuration space $\mathcal{C}$ equals $\mathbb{R}$. The wave functions of quantum mechanics then are L_2 functions over $\mathbb{R}$ w.r.t. the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}$. In the Schrödinger representation, the position and momentum variables x and p turn into self-adjoint multiplication and derivation operators fulfilling $[\widehat{x}, \widehat{p}] = \mathrm{i}$. They generate weakly continuous one-parameter subgroups of unitaries: multiplication operators $\mathrm{e}^{\mathrm{i}\sigma\widehat{x}} = \mathrm{e}^{\mathrm{i}\sigma x}\cdot$ and pull-backs $\mathrm{e}^{\mathrm{i}\lambda\widehat{p}} = L_\lambda^*$ of translation operators on $\mathcal{C} = \mathbb{R}$.

Although we presented the quantization procedure above like an unrevocable fact, there are many other options to quantize classical mechanics. The easiest way to see this is to exchange the rôles of x and p. There we end up with the Heisenberg picture of quantum mechanics with $\widehat{p}$ being a multiplication and $\widehat{x}$ a differential operator. Nevertheless, the Heisenberg and Schrödinger pictures are still unitarily equivalent, i.e. physically indistinguishable. Indeed, even more, the Stone-von Neumann theorem tells us that assuming continuity and irreducibility, all "pictures" of quantum mechanics are equivalent: Each pair (U, V) of unitary representations of $\mathbb{R}$ on some Hilbert space $\mathfrak{H}$ satisfying the commutation relations

$$U(\sigma)V(\lambda) \quad = \quad \mathrm{e}^{\mathrm{i}\sigma\lambda}\, V(\lambda)U(\sigma) \tag{1.1}$$

for all $\sigma, \lambda \in \mathbb{R}$, is equivalent to multiples of the Schrödinger representation above [14]. Hence, assuming irreducibility, we get the desired uniqueness. It should be emphazised that the commutation relations (1.1) directly follow from $U(\sigma) = \mathrm{e}^{\mathrm{i}\sigma X}$, $V(\lambda) = \mathrm{e}^{\mathrm{i}\lambda P}$ and $[X, P] = \mathrm{i}$, i.e., they do not use any relations induced by special representations.

Finally, however, note that dropping the continuity assumption admits other, non-equivalent representations. One of them is given by almost-periodic functions leading to the Bohr compactification of the real line. The Hilbert space basis is given by $\{|x\rangle \mid x \in \mathbb{R}\}$, and the operators U and V act by $U(\sigma)|x\rangle = \mathrm{e}^{\mathrm{i}\sigma x}|x\rangle$ and $V(\lambda)|x\rangle = |x + \lambda\rangle$. Of course, V is not continuous. Hence, $\widehat{p}$ is not defined, but the operator $V(\lambda)$ corresponding to $\mathrm{e}^{\mathrm{i}\lambda p}$ only. We remark that this representation reappears in loop quantum cosmology yielding to a resolution of the big bang singularity [7].

2 Configuration Space of Quantum Geometry

Let us now focus to (pure) gauge field theories. There the configuration space $\mathcal{C}$ consists of all smooth connections (modulo gauge transforms) in a principal fibre bundle $P(M, \mathbf{G})$ with M being some manifold and $\mathbf{G}$ being some structure Lie group. Here, we will assume that M is at least two-dimensional and that $\mathbf{G}$ is connected and compact. For canonical gravity using Ashtekar variables [1], e.g., M is some Cauchy slice and $\mathbf{G}$ equals $SU(2)$. Aiming at a functional-integral description of quantum theory, one needs some measure[1] on $\mathcal{C}$. In general, however, the structure of $\mathcal{C}$ is too complicated to allow for a rigorous measure theory describing non-pathological measures. Therefore, in parallel to the experiences known from the Wiener integral, it is reasonable to compactify $\mathcal{C}$.

2.1 Compactification

Ashtekar et al. [2, 4, 3] successfully implemented the compactification strategy. Their crucial idea was to use parallel transports instead of connections. In fact, a connection is uniquely determined by its parallel transports along all (sufficiently smooth) paths in M. Now, one considers the holonomy algebra[2] $\mathcal{HA}$, which is a subset of the bounded

[1]From now on, all measures are assumed to be normalized, regular, and Borel.

[2]Note that in [2] the holonomy algebra denotes the algebra generated by all Wilson loop variables, giving the gauge invariant functions only. Here, we also include gauge variant functions.

functions on the space $\mathcal{A} \equiv \mathcal{C}$ of smooth connections. $\mathcal{HA}$ is the (unital) $*$-subalgebra of $C_{\text{bound}}(\mathcal{A})$ generated by all matrix elements $T_{\gamma,\phi,m,n} := \phi(h_\gamma)^m_n$ of parallel transports h_γ, where γ runs over all paths in M, ϕ runs over all (equivalence classes of) irreducible representations of $\mathbf{G}$, and m and n over all the corresponding matrix indices. The space $\overline{\mathcal{A}}$ of generalized (or, distributional) connections is now defined to be the spectrum of the completion of the holonomy algebra. Since $\overline{\mathcal{HA}}$ is unital abelian, $\overline{\mathcal{A}}$ is compact Hausdorff.

Equivalently, $\overline{\mathcal{A}}$ can be described using projective limits. First, observe that the elements of $\overline{\mathcal{A}}$ are one-to-one with the homomorphisms from the groupoid[3] $\mathcal{P}$ of paths in M to the structure group $\mathbf{G}$. In fact, each $h \in \operatorname{Hom}(\mathcal{P}, \mathbf{G})$ defines a multiplicative functional $h_{\overline{A}}$ on $\overline{\mathcal{HA}}$ via $h_{\overline{A}}(T_{\gamma,\phi,m,n}) := \phi(h(\gamma))^m_n$ implying $h_{\overline{A}} \in \overline{\mathcal{A}}$. Now, any finite graph $\boldsymbol{\gamma}$ in M defines a continuous projection $\pi_{\boldsymbol{\gamma}} : \overline{\mathcal{A}} \longrightarrow \operatorname{Hom}(\mathcal{P}_{\boldsymbol{\gamma}}, \mathbf{G}) \cong \mathbf{G}^{\#\boldsymbol{\gamma}}$ via

$$\pi_{\boldsymbol{\gamma}}(h_{\overline{A}}) \;:=\; h_{\overline{A}}|_{\mathcal{P}_{\boldsymbol{\gamma}}} \;\widehat{=}\; h_{\overline{A}}(\boldsymbol{\gamma}) \;\equiv\; \big(h_{\overline{A}}(\gamma_1), \ldots, h_{\overline{A}}(\gamma_{\#\boldsymbol{\gamma}})\big),$$

where $\mathcal{P}_{\boldsymbol{\gamma}}$ denotes the paths in $\boldsymbol{\gamma}$. Note that the edges $\gamma_1, \ldots, \gamma_{\#\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ freely generate $\mathcal{P}_{\boldsymbol{\gamma}}$. Using the natural subgraph relation and defining $\pi^{\boldsymbol{\delta}}_{\boldsymbol{\gamma}} : \operatorname{Hom}(\mathcal{P}_{\boldsymbol{\delta}}, \mathbf{G}) \longrightarrow \operatorname{Hom}(\mathcal{P}_{\boldsymbol{\gamma}}, \mathbf{G})$ for $\boldsymbol{\gamma} \leq \boldsymbol{\delta}$ again by restriction, we get a projective system over the set of all finite graphs in M, whose projective limit $\varprojlim_{\boldsymbol{\gamma}} \operatorname{Hom}(\mathcal{P}_{\boldsymbol{\gamma}}, \mathbf{G})$ is $\overline{\mathcal{A}}$ again.

2.2 Ashtekar-Lewandowski Measure

Compactness opens the door to many far-reaching theorems in measure theory, in particular, on projective limits. In general, given a measure μ on a projective limit $X := \varprojlim_a X_a$, we may always push-forward this to all X_a using the canonical projections $\pi_a : X \longrightarrow X_a$. Of course, there are certain compatibility relations among different constituents: If π^b_a projects X_b to X_a then $(\pi^b_a)_*(\pi_b)_*\mu$ equals $(\pi_b)_*\mu$. For a directed projective limit of compact Hausdorff spaces, however, the Riesz-Markov theorem establishes also the other way round. For each sequence μ_a of measures on X_a that fulfill the compatibility relations $(\pi^b_a)_*\mu_b = \mu_a$ for all $a \leq b$, there is a unique measure μ on X with $(\pi_a)_*\mu = \mu_a$ for all a.

Using this general theorem, it is rather easy to define measures on $\overline{\mathcal{A}}$. The most obvious choice gives the Ashtekar-Lewandowski measure μ_0 [4] whose relevance will become clear below. Here, one simply demands that the measure on $\operatorname{Hom}(\mathcal{P}_{\boldsymbol{\gamma}}, \mathbf{G}) \cong \mathbf{G}^{\#\boldsymbol{\gamma}}$ is the Haar measure for each $\boldsymbol{\gamma}$. If the set of all finite graphs is directed, the compatibility conditions are fulfilled and guarantee the existence and uniqueness of μ_0. The directedness is given if we assume all paths and graphs to be piecewise analytic. It is no longer given for smooth paths. There it may happen that two graphs have infinitely many intersection points without sharing a full segment, whence there is no third graph containing both graphs. Although it is in principle possible to circumvent this problem [9], we will restrict ourselves to piecewise analytic paths from now on.

2.3 Spin Networks

Having now $\mathfrak{H}_0 := L_2(\overline{\mathcal{A}}, \mu_0)$ as a candidate for the kinematical Hilbert space, we still have to look for a basis or, at least, some reasonable generating system of this space. As in the case of measures, the problem is solved by focussing first on the graph level and then lifting it to the continuum. In fact, bases for $L_2(\mathbf{G}^n, \mu_{\text{Haar}})$ are given by the

[3] The groupoid structure is induced by the standard concatenation of paths modulo reparametrization and deletion/insertion of immediate retracings.

Peter-Weyl theorem. Each one contains just all tensor products of matrix elements of irreducible representations of the n group factors. Combining this with the projections $\pi_{\boldsymbol{\gamma}}$, we get the spin network functions [5][4]

$$\bigotimes_k T_{\gamma_k,\phi_k,m_k,n_k} \;\equiv\; \bigotimes_k (\phi_k)^{m_k}_{n_k} \circ \pi_{\gamma_k} \;:\; \overline{\mathcal{A}} \longrightarrow \mathbb{C}.$$

Here, $\boldsymbol{\gamma} = \{\gamma_1, \ldots, \gamma_{\#\boldsymbol{\gamma}}\}$ is a graph, each ϕ_k a nontrivial irreducible representation of $\mathbf{G}$, and each m_k and each n_k a matrix index. The set of all spin network functions generates $\mathfrak{H}_0$. However, it does not form a basis, since two spin network functions that correspond, e.g., to graphs where one graph is a refinement of the other, need not be orthogonal. But, at least, the trivial spin network function, i.e., the constant function, is orthogonal to all the others.

2.4 Semianalytic Diffeomorphisms

Until now, only general gauge theory ingredients have been implemented. Now, we are going to consider the most important difference between quantum gauge field theories and quantum geometry – the diffeomorphism invariance. Fortunately, within the loop formalism, this task is straightforward. The action of diffeomorphisms on M lifts naturally to an action on the sets of paths and graphs, whence to an action on $\overline{\mathcal{A}}$ and finally on $C(\overline{\mathcal{A}})$ as well. For instance, let there be given a cylindrical function, i.e. a function $f \in C(\overline{\mathcal{A}})$ which equals $f_{\boldsymbol{\gamma}} \circ \pi_{\boldsymbol{\gamma}}$ for some graph $\boldsymbol{\gamma}$ and some $f_{\boldsymbol{\gamma}} \in C(\pi_{\boldsymbol{\gamma}}(\overline{\mathcal{A}}))$. Then $\alpha_\varphi(f_{\boldsymbol{\gamma}} \circ \pi_{\boldsymbol{\gamma}}) = f_{\boldsymbol{\gamma}} \circ \pi_{\varphi(\boldsymbol{\gamma})}$, with α_φ denoting the action of the diffeomorphism φ on $C(\overline{\mathcal{A}})$. One shows easily that μ_0 is diffeomorphism invariant, whence the action of diffeomorphisms is even unitary on $\mathfrak{H}_0$.

Of course, we can only consider diffeomorphisms that preserve the piecewise analyticity of paths and graphs. Smooth diffeomorphisms, in general, do not meet this requirement. Analytic ones, on the other hand, are too restrictive from the physical point of view. In fact, gravity is a local theory, meaning it is invariant w.r.t. diffeomorphisms being the identity outside some subset of M. But, analyticity is nonlocal: Changing an analytic object locally, modifies it globally. In other words, we are looking for some intermediate kind of diffeomorphisms reconciling locality and analyticity. This is indeed possible working in the semianalytic category [12, 13, 6] what we will always do in the following. Recall that semianalytic sets are stratified by analytic manifolds, whence semianalytic diffeomorphisms only need to be analytic on each single stratum.

3 Weyl Algebra of Quantum Geometry

In quantum mechanics, the Schrödinger representation assigns multiplication operators to the exponentiated position operators and pull-backs of translation operators to the exponentiated momentum operators. In quantum geometry, since we are dealing with fields, we do not only exponentiate the operators, but also smear them. In fact, parallel transports are exponentiated connections smeared along paths being one-dimensional objects. The corresponding momenta in quantum geometry (or, more specific, in canonical gravity) are densitized dreibein fields. They are now exponentiated after getting smeared along one-*co*dimensional objects, namely (semi-)analytic submanifolds. While the parallel transports (or, to be precise, their matrix elements) act naturally by multiplication

[4] Note again that, originally, spin network functions have been defined a little bit differently in order to implement gauge invariance.

operators on $\mathfrak{H}_0$, the exponentiated and smeared dreibein field operators will turn into unitary pull-backs of translation operators on $\overline{\mathcal{A}}$.

3.1 Weyl Operators

Let now S be some oriented analytic hypersurface and let $g \in \mathbf{G}$. We define the intersection function $\sigma_S(\gamma)$ to be $+1$ (-1) if the path γ starts at S, such that $\dot{\gamma}$ is non-tangent to S and such that some initial path of γ lies fully above (below) S; otherwise $\sigma_S(\gamma)$ is 0. Above and below, of course, refer to the orientation of S. Next, observe that due to the semianalyticity of paths and hypersurfaces, each path can be decomposed into a finite number of paths whose interior is either fully contained in S (internal path) or disjoint to S (external path). One now checks quite easily [8] that there is a unique map $\Theta_g^S : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}$, such that[5]

$$h_{\Theta_g^S(\overline{A})}(\gamma) \quad = \quad g^{\sigma_S(\gamma)}\, h_{\overline{A}}(\gamma)\, g^{-\sigma_S(\gamma^{-1})}$$

for all external and all internal $\gamma \in \mathcal{P}$ and for all $\overline{A} \in \overline{\mathcal{A}}$. The map Θ_g^S is a homeomorphism. It even preserves the Ashtekar-Lewandowski measure, due to the translation invariance of the Haar measure.

The Weyl operator w_g^S is now the pull-back of Θ_g^S. It becomes an isometry on $C(\overline{\mathcal{A}})$ and, by μ_0-invariance, a unitary operator on $\mathfrak{H}_0 = L_2(\overline{\mathcal{A}}, \mu_0)$. The subset of $\mathcal{B}(\mathfrak{H}_0)$ generated by all Weyl operators $\{w_g^S\}$ is denoted by $\mathcal{W}$. We remark that the Weyl operator of the disjoint union of hypersurfaces equals the product of the (mutually commuting) Weyl operators of the single hypersurfaces. This way, it is natural to extend the notion of Weyl operators even to semianalytic submanifolds S having at least codimension 1. For instance, the Weyl operator for an equator (i.e., a one-dimensional circle in a three-dimensional space) can be seen as the Weyl operator of the full sphere times the inverses of the Weyl operators corresponding to the upper and lower hemisphere. Therefore, w.l.o.g., $\mathcal{W}$ is always assumed to contain also the semianalytic subsets of M having at least codimension 1.

The diffeomorphisms act covariantly on $\mathcal{W}$ via $\alpha_\varphi \circ w_g^S \circ \alpha_\varphi^{-1} \equiv \alpha_\varphi(w_g^S) = w_g^{\varphi(S)}$, as one checks immediately.

3.2 Weyl Algebra

The C^*-subalgebra $\mathfrak{A}$ of $\mathcal{B}(L_2(\overline{\mathcal{A}}, \mu_0))$, generated by $C(\overline{\mathcal{A}})$ and $\mathcal{W}$, is called **Weyl algebra of quantum geometry**. Its natural representation on $L_2(\overline{\mathcal{A}}, \mu_0)$ will be denoted by π_0.

Sometimes, we will consider the C^*-subalgebra of $\mathcal{B}(L_2(\overline{\mathcal{A}}, \mu_0))$ generated by the Weyl algebra $\mathfrak{A}$ and the diffeomorphism group $\mathcal{D}$. It will be denoted by $\mathfrak{A}_{\mathrm{Diff}}$. One immediately sees that $\mathcal{D}$ acts covariantly on $\mathfrak{A}$.

3.3 Irreducibility

In this subsection, we are going to prove the irreducibility of $\mathfrak{A}$. For this, let $f \in \mathfrak{A}'$. First, by $C(\overline{\mathcal{A}}) \subseteq \mathfrak{A}$, we have $\mathfrak{A}' \subseteq C(\overline{\mathcal{A}})' = L_\infty(\overline{\mathcal{A}}, \mu_0)$. Second, by unitarity of Weyl operators w, we have $f = w^* \circ f \circ w = w^*(f)$. Therefore, we have $\langle T, f\rangle = \langle T, w^*(f)\rangle =$

[5] Usually, instead of g being just an element of $\mathbf{G}$, it denotes a function from S to $\mathbf{G}$, encoding the smearing of the flux. We will not use non-constant smearings here; we skip this possibility.

$\langle w(T), f\rangle$ for every spin network function T. Each nontrivial T may be decomposed into $T = (T_{\gamma,\phi,m,n})\, T'$ with some edge γ and nontrivial ϕ, where T' is a (possibly trivial) spin network function.

- If ϕ is abelian, choose some hypersurface S intersecting γ, but no edge used for T'. Then $w_g^S(T) = \phi(g^2)\, T$ for all $g \in \mathbf{G}$, whence $\langle T, f\rangle = \langle w_g^S(T), f\rangle = \overline{\phi(g^2)}\, \langle T, f\rangle$. Since ϕ is nontrivial, there is some $g \in \mathbf{G}$ with $\phi(g^2) \neq 1$. Hence, $\langle T, f\rangle = 0$.

- If ϕ is nonabelian, then $\operatorname{tr}\phi$ has a zero [11]. Since square roots exist in any compact connected Lie group, there is a $g \in \mathbf{G}$ with $\operatorname{tr}\,\phi(g^2) = 0$. Choose now infinitely many mutually disjoint surfaces S_i intersecting γ, but no edge used for T'. A straightforward calculation yields for $i \neq j$

$$\langle w_g^{S_i}(T), w_g^{S_j}(T)\rangle \quad = \quad \left|\frac{\operatorname{tr}\,\phi(g^2)}{\dim\phi}\right|^2 \quad = \quad 0.$$

  Now, $\langle w_g^{S_i}(T), f\rangle = \langle T, f\rangle = \langle w_g^{S_j}(T), f\rangle$ implies $\langle T, f\rangle = 0$ again.

Altogether, f is constant. Hence, $\mathfrak{A}'$ consists of scalars only.

4 Uniqueness Theorem

The natural representation π_0 of $\mathfrak{A}$ is not only irreducible, but also regular, i.e., it is weakly continuous w.r.t. the Weyl operator smearings g. Moreover, π_0 is diffeomorphism invariant, i.e., there is a diffeomorphism invariant vector in $\mathfrak{H}_0$ (the constant function) and the diffeomorphisms act covariantly on $\mathfrak{A}$. The fundamental question now is how far these properties already determine π_0 among the C^*-algebra representations of $\mathfrak{A}$. Fairly uniquely, as we will learn from the following theorem.

4.1 Theorem

Let $\dim M \geq 3$, let $\mathbf{G}$ be nontrivial, and let all the paths, hypersurfaces, and diffeomorphisms be semianalytic. Assume that all the hypersurfaces used for the definition of $\mathcal{W}$ are widely[6] triangulizable. Let now $\pi : \mathfrak{A}_{\text{Diff}} \longrightarrow \mathcal{B}(\mathfrak{H})$ be some regular C^*-algebra representation of $\mathfrak{A}_{\text{Diff}}$ on some Hilbert space $\mathfrak{H}$, which has some diffeomorphism invariant vector being cyclic for $\pi|_{\mathfrak{A}}$. If the diffeomorphisms even act naturally[7], then $\pi|_{\mathfrak{A}}$ is unitarily equivalent to π_0.

4.2 Sketch of Proof

The proof will consist of three main steps: First, by general C^*-algebra arguments, we have $\pi|_{C(\overline{\mathcal{A}})} = \bigoplus_\nu \pi_\nu$, where each π_ν is the canonical representation of $C(\overline{\mathcal{A}})$ on $L_2(\overline{\mathcal{A}}, \mu_\nu)$ for some measure μ_ν. Second, we prove that π_ν equals $\pi_0|_{C(\overline{\mathcal{A}})}$ for some ν using diffeomorphism invariance and regularity. Finally, from naturality, diffeomorphism invariance and cyclicity we deduce that the directed sum above consists of just a single component

[6] A triangulation (K, f) is called wide iff for every $\sigma \in K$ there is some open chart in M containing the closure of $f(\sigma)$ and mapping it to a simplex in that chart.

[7] The definition of naturality will be provided in the sketched proof below.

and that $\pi|_{\mathfrak{A}} = \pi_0$. Let us now sketch the final steps 2 and 3. The full proof may be found in [10].

For step 2, let us assume for simplicity that $\mathbf{G}$ is abelian. Fix $\varepsilon > 0$, and let $f = h_\gamma(\cdot)^n T$ be some nontrivial spin network function, i.e., γ and the graph underlying T form a graph again and $n \neq 0$. Assume $\langle \mathbf{1}_0, \pi(f)\mathbf{1}_0\rangle_{\mathfrak{H}} \neq 0$ for some (cyclic and) diffeomorphism invariant vector $\mathbf{1}_0 \in \mathfrak{H}$. Define for some "cubic" hypersurface S

$$w_t := w^S_{\mathrm{e}^{\mathrm{i}t/2}} \quad \text{and} \quad v_t := \frac{1}{2^m}\sum_{k=1}^{2^m} \alpha_{\varphi_k}(w_t).$$

Here, each φ_k is a diffeomorphism winding γ, such that it has exactly m punctures with S. Each k corresponds to a sequence of m signs $+$ or $-$. These signs correspond to the relative orientations of S and $\varphi_k(\gamma)$ at the m punctures. Then

$$v_t(f) \;=\; \Bigl(\frac{\mathrm{e}^{int} + \mathrm{e}^{-int}}{2}\Bigr)^m \cdot f$$

and

$$\|(v_t - \mathrm{e}^{-\frac{1}{2}m(nt)^2})\, f\|_\infty \;\leq\; O(m(nt)^4)\,\|f\|_\infty.$$

Hence, for any small t there is some $m = m(nt,\varepsilon)$ with

$$\begin{aligned}
\varepsilon \;&<\; O(m(nt)^2)\,|\langle \mathbf{1}_0, \pi(f)\mathbf{1}_0\rangle_{\mathfrak{H}}| - O(m(nt)^4)\,\|f\|_\infty \\
&\leq\; |(1-\mathrm{e}^{-\frac{1}{2}m(nt)^2})\langle \mathbf{1}_0, \pi(f)\mathbf{1}_0\rangle_{\mathfrak{H}}| - |\langle \mathbf{1}_0, \pi[(v_t - \mathrm{e}^{-\frac{1}{2}m(nt)^2})f]\mathbf{1}_0\rangle_{\mathfrak{H}}| \\
&\leq\; |\langle \mathbf{1}_0, \pi[(v_t - \mathbf{1})f]\mathbf{1}_0\rangle_{\mathfrak{H}}|.
\end{aligned}$$

Now, for each small t there is a diffeomorphism φ with

$$\begin{aligned}
\varepsilon \;&\leq\; |\langle \mathbf{1}_0, \pi[(w_t - \mathbf{1})(\alpha_\varphi(f))]\mathbf{1}_0\rangle_{\mathfrak{H}}| \\
&\leq\; 2\,\|\mathbf{1}_0\|_{\mathfrak{H}}\,\|(\pi(w_t) - \mathbf{1})\mathbf{1}_0\|_{\mathfrak{H}}\,\|\pi(\alpha_\varphi(f))\|_{\mathcal{B}(\mathfrak{H})} \\
&=\; 2\,\|\mathbf{1}_0\|_{\mathfrak{H}}\,\|(\pi(w_t) - \mathbf{1})\mathbf{1}_0\|_{\mathfrak{H}}\,\|f\|_\infty,
\end{aligned}$$

by diffeomorphism invariance. The final term, however, does not depend on φ, whence by regularity it goes to zero for $t \to 0$ giving a contradiction. Hence, $\langle \mathbf{1}_0, \pi(f)\mathbf{1}_0\rangle_{\mathfrak{H}} = 0$ for $n \neq 0$ implying $\pi|_{C(\overline{\mathcal{A}})} \cong \pi_0$.

For step 3, let w be some Weyl operator assigned to an open ball or simplex, possibly having higher codimension. Observe that, for w commuting with α_φ, we have

$$\langle \pi(\alpha_\varphi(f))\mathbf{1}_0, \pi(w)\mathbf{1}_0\rangle_{\mathfrak{H}} \;=\; \langle \pi(f)\mathbf{1}_0, \pi(w)\mathbf{1}_0\rangle_{\mathfrak{H}}.$$

Next, for nonconstant spin-network functions f, choose diffeomorphisms φ_i, commuting with w, such that

$$\delta_{ij} \;=\; \langle \alpha_{\varphi_i}(f), \alpha_{\varphi_j}(f)\rangle_{\mathfrak{H}_0} \;\equiv\; \langle \pi(\alpha_{\varphi_i}(f))\mathbf{1}_0, \pi(\alpha_{\varphi_j}(f))\mathbf{1}_0\rangle_{\mathfrak{H}}.$$

(This, however, is not always possible causing some technical difficulties that will, nevertheless, be ignored in the present article.) Consequently, we have

$$0 \;=\; \langle \pi(f)\mathbf{1}_0, \pi(w)\mathbf{1}_0\rangle_{\mathfrak{H}} \;=\; \langle f, P_0\pi(w)\mathbf{1}_0\rangle_{\mathfrak{H}_0},$$

with P_0 being the canonical projection from $\mathfrak{H}$ to $\overline{\pi(C(\overline{\mathcal{A}}))\,\mathbf{1}_0} \cong L_2(\overline{\mathcal{A}},\mu_0) \equiv \mathfrak{H}_0$. Now, $P_0\pi(w)\mathbf{1}_0 = c(w)\,\mathbf{1}_0$ with $c(w) \in \mathbb{C}$, whence $(\mathbf{1} - P_0)\pi(w)\mathbf{1}_0$ generates $L_2(\overline{\mathcal{A}},\mu_0)$. The naturality of π w.r.t. the action of diffeomorphisms implies that $\pi(w)\mathbf{1}_0$ is diffeoinvariant itself.[8] Since S is assumed to be a ball or simplex (with lower dimension than M), there is a (semianalytic) diffeomorphism mapping S to itself, but inverting its orientation. Now, we have

$$\pi(w)^2 \mathbf{1}_0 \;=\; \pi(w)\pi(\alpha_\varphi)\pi(w)^*\pi(\alpha_\varphi)^* \mathbf{1}_0 \;=\; \mathbf{1}_0$$

implying $\pi(w)\mathbf{1}_0 = \mathbf{1}_0$ by taking the square root of the smearing. The proof furnishes using cyclicity and triangulizability.

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[8] A representation π is called natural w.r.t. the action of diffeomorphisms iff, for each decomposition $\bigoplus_\nu \pi_\nu$ into cyclic components of $\pi|_{C(\overline{\mathcal{A}})}$, the diffeomorphism invariance of the $C(\overline{\mathcal{A}})$-cyclic vector for π_{ν_1} implies that for π_{ν_2}, provided the measures μ_1 and μ_2 underlying these representations coincide.

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Orbits of triples in the Shilov boundary of a bounded symmetric domain

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Abstract

Let $\mathcal{D}$ be a bounded symmetric domain of tube type, S its Shilov boundary, and G the neutral component of its group of biholomorphic transforms. We classify the orbits of G in the set $S \times S \times S$.

1 The problem

It is well-known that any bounded symmetric domain $\mathcal{D}$ in a finite-dimensional complex vector space can be realized as the open unit ball

$$\mathcal{D} = \{z \in V \; \|z\| < 1\} \tag{1}$$

of a normed complex vector space V. Such a unit ball is *symmetric* if and only if the group $\mathrm{Aut}(\mathcal{D})$ of biholomorphic transforms of $\mathcal{D}$ acts transitively, i.e., if $\mathcal{D}$ is homogeneous. Let $G := \mathrm{Aut}(\mathcal{D})_0$ be the identity component of $\mathrm{Aut}(\mathcal{D})$ and S be the Shilov boundary of $\mathcal{D}$. The action of any element of G extends to a neighborhood of $\overline{\mathcal{D}}$, and hence G acts on S. It is well known that this action is transitive. The main result we are reporting on is a classification of the G-orbits in the set $S \times S \times S$ of triples in S, when $\mathcal{D}$ is of *tube type*. The action of G on $S \times S$ can be easily studied as an application of Bruhat theory, and the description of the orbits is the same, whether $\mathcal{D}$ is of tube type or not. But for triples there is a drastic difference between tube type domains and non tube type domains. In the first case, there is a finite number of orbits in $S \times S \times S$, whereas there are an infinite number of orbits for a non tube type domain.

2 Jordan triples, Jordan frames, and polydiscs

In the following we shall always assume that $\mathcal{D}$ is realized as a unit ball as in (1). Then $\mathcal{D}$ is said to be *reducible* if $V = V_1 \times V_2$ with the norm on V satisfying $\|(v_1, v_2)\| = \max(\|v_1\|, \|v_2\|)$, so that $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$, where $\mathcal{D}_j$ is the unit ball in V_j.

Any $\mathcal{D}$ can be written as a product or irreducible domains, and since G and S decompose accordingly as products, we assume in the following that $\mathcal{D}$ is irreducible.

Then G is a finite-dimensional simple real Lie group acting transitively on $\mathcal{D}$, the stabilizer K of $0 \in \mathcal{D}$ is maximal compact and $\mathcal{D} \cong G/K$ is a non-compact Riemannian

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Keywords: bounded symmetric domain, tube type domain, Shilov boundary, face, Maslov index, flag manifold, Jordan triples

Subject classification: 32M15, 53D12, 22F30

symmetric space of hermitian type. We consider the Lie algebra $\mathfrak{g}$ of G as realized by vector fields on $\mathcal{D}$. Since all these vector fields are polynomial of degree ≤ 2, they extend to all of V. Moreover, $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathbf{L}(K)$ consists of linear vector fields and $\mathfrak{p}$ consists of even vector fields. Since the evaluation map $\mathfrak{p} \to V \cong T_0(\mathcal{D}), X \mapsto X(0)$ is a linear isomorphism, each element of $\mathfrak{p}$ can be written as

$$X_v(z) = v + Q(z)(v),$$

where $Q(z)(v)$ is quadratic in z and real linear in v. Polarization now leads to the triple product

$$\{\cdot,\cdot,\cdot\}\, V^3 \to V$$

which is uniquely determined by

$$Q(z)(v) = \{z, v, z\} \quad \text{and} \quad \{a, b, c\} = \{c, b, a\} \quad \text{for} \quad a, b, c, z, v \in V.$$

By definition, the triple product is linear in the first and third argument. It is antilinear in the second argument.

An element $e \in V$ is called a *tripotent* if $\{e, e, e\} = e$ and two tripotents e, f are said to be *orthogonal* if $\{e, e, f\} = 0$, which turns out to define a symmetric relation. The sum $e + f$ of two non-zero orthogonal tripotents is a tripotent, called *decomposable*. A *Jordan frame* is a maximal tuple $(c_1, \ldots, c_r)$ of pairwise orthogonal indecomposable tripotents. Their number r is called the *rank of* $\mathcal{D}$.

It is a well-known fact that for each Jordan frame $(c_1, \ldots, c_r)$ with $E := \mathrm{span}\{c_1, \ldots, c_r\}$ the intersection

$$\mathcal{D} \cap E = \Big\{ \sum_{j=1}^{r} \zeta_j c_j \,|\zeta_j| < 1, 1 \leq j \leq r \Big\} \cong \Delta^r$$

is an r-dimensional polydisc, i.e., the unit ball in $(\mathbb{C}^r, \|\cdot\|_\infty)$. The converse is less obvious ([CN05]):

Theorem 2.1. *If $E \subseteq V$ is an r-dimensional subspace for which $E \cap \mathcal{D}$ is a polydisc, then there exists a Jordan frame $(c_1, \ldots, c_r)$ in V spanning E.*

3 Faces and orbits

It is easy to describe the G-orbit structure on $\overline{\mathcal{D}}$ in terms of a Jordan frame: There are $r + 1$ orbits which are represented by the tripotents

$$e_k := c_1 + \ldots + c_k, \quad k = 0, \ldots, r.$$

Here $e_0 = 0$, $G.e_0 = \mathcal{D}$, and $G.e_k = S$ is the Shilov boundary of $\mathcal{D}$. We may therefore define a *rank function*

$$\mathrm{rank}\ \overline{\mathcal{D}} \to \{0, \ldots, r\}, \quad g.e_k \mapsto k,$$

classifying the G-orbits in $\overline{\mathcal{D}}$.

Our next goal is to extend this rank function to tuples of elements in $\overline{\mathcal{D}}$, which is done by using faces of the compact convex set $\overline{\mathcal{D}}$. For a subset $M \subseteq \overline{\mathcal{D}}$ we write $\mathrm{Face}(M) \subseteq \overline{\mathcal{D}}$

for the *face generated by M*, i.e., the intersection of all faces of $\overline{\mathcal{D}}$ containing it. Using the result that the faces of $\overline{\mathcal{D}}$ coincide with the closures of the holomorphic arc components of $\overline{\mathcal{D}}$, we see that the group G acts on the set $\mathcal{F}(\overline{\mathcal{D}})$ of faces of $\overline{\mathcal{D}}$. There are precisely $r+1$ orbits, represented by the faces $\mathrm{Face}(e_k)$, $k = 0, \ldots, r$. As for $\overline{\mathcal{D}}$, this leads to a G-invariant rank function

$$\mathrm{rank}\ \mathcal{F}(\overline{\mathcal{D}}) \to \{0, \ldots, r\}, \quad g.\mathrm{Face}(e_k) = \mathrm{Face}(g.e_k) \mapsto k,$$

classifying the G-orbits.

This picture provides a nice connection between the convex geometry of $\overline{\mathcal{D}}$ and its complex geometric properties. In particular, the Shilov boundary S coincides with the set of extreme points, i.e., the one-point faces of $\overline{\mathcal{D}}$.

From the rank function for faces, we immediately obtain integral invariants for the diagonal G-action on the product sets $\overline{\mathcal{D}}^m$:

$$\overline{\mathcal{D}}^m \to \{0, \ldots, r\}, \quad (z_1, \ldots, z_m) \mapsto \mathrm{rank}(\mathrm{Face}(z_1, \ldots, z_m)).$$

4 The Diagonalization Theorem and its consequences

The main result of Section 1 in [CN05] is a classification of the G-orbits in the set $\overline{\mathcal{D}}^2$ of pairs which are *transversal* in the sense that $\mathrm{rank}\,\mathrm{Face}(x, y) = 0$, i.e., x and y do not lie in a common proper face of $\overline{\mathcal{D}}$. The main tool for the classification of G-orbits in $S \times S \times S$ (for tube type domains) is the characterization of this relation in Jordan theoretic terms: it is equivalent to quasi-invertibility ([CN05, Th. 2.6]). This characterization is also valid for non tube type domains. This fact is used to setup an inductive proof of the

Theorem 4.1 (Diagonalization Theorem). *If $\mathcal{D}$ is a bounded symmetric domain of tube type and $\Delta^r \subseteq \mathcal{D}$ is a polydisc of maximal rank r (hence given by a Jordan frame), then every triple in S is conjugate to a triple in the Shilov boundary T of Δ^r.*

The assumption that $\mathcal{D}$ is of tube type is essential in the preceding theorem because it is invalid for all non-tube type domains.

This reduces the classification of G-orbits in $S\times S\times S$ to the description of intersections of these orbits with the $3r$-dimensional torus T^3. This is fully achieved by assigning a 5-tuple $(r_0, r_1, r_2, r_3, \iota)$ of integer invariants to each orbit and by showing that triples with the same invariant lie in the same orbit. The first four components of this 5-tuple are

$$\begin{aligned}&(r_0, \ldots, r_3) = \\ &\quad (\mathrm{rank}\,\mathrm{Face}(x_1, x_2, x_3), \mathrm{rank}\,\mathrm{Face}(x_1, x_2), \mathrm{rank}\,\mathrm{Face}(x_2, x_3), \mathrm{rank}\,\mathrm{Face}(x_1, x_3)).\end{aligned} \tag{4.1}$$

The fifth component is the Maslov index $\iota(x_1, x_2, x_3)$, an integer invariant taking values in $\{-r, \ldots, r\}$ (cf. [CØ01], [Cl04a/b]). If all pairs (x_1, x_2), (x_2, x_3) and (x_3, x_1) are transversal, then all r_i vanish, which implies that the G-orbits in the set of transversal triples are classified by the Maslov index.

Theorem 4.2 (Classification of orbits). *Let $\mathcal{D}$ be a bounded symmetric domain of tube type. Then the five integers $(r_0, r_1, r_2, r_3, \iota)$ separate the orbits of G in $S \times S \times S$, and a 5-tuple $(r_0, r_1, r_2, r_3, \iota) \in \mathbb{Z}^5$ arises from some orbit if and only if*

(P1) $0 \leq r_0 \leq r_1, r_2, r_3 \leq r$.
(P2) $r_1 + r_2 + r_3 \leq r + 2r_0$.
(P3) $|\iota| \leq r + 2r_0 - (r_1 + r_2 + r_3)$.
(P4) $\iota \equiv r + r_1 + r_2 + r_3 \bmod 2$.

A special case of this theorem was known before: If $\mathcal{D}$ is the Siegel domain (the unit ball in the space of complex symmetric matrices $\mathrm{Sym}_r(\mathbb{C})$), then the group G is the projective symplectic group $\mathrm{PSp}_{2r}(\mathbb{R}) := \mathrm{Sp}_{2r}(\mathbb{R})/\{\pm\mathbf{1}\}$, and the Shilov boundary of $\mathcal{D}$ can be identified with the Lagrangian manifold (the set of Lagrangian subspaces of $\mathbb{R}^{2r}$). Then the orbits of triples of Lagrangians have been described (see [KS90, p.492]), using linear symplectic algebra techniques.

As a byproduct of Theorem 2, we also obtain the following axiomatic characterization of the Maslov index, which actually was our original motivation for the project:

Theorem 4.3. *The Maslov index is characterized by the following properties:*
(M1) *It is invariant under the group G.*
(M2) *It is an alternating function with respect to any permutation of the three arguments.*
(M3) *It is additive in the sense that if $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$, so that $S = S_1 \times S_2$, then*

$$\iota_S(x, y, z) = \iota_S((x_1, x_2), (y_1, y_2), (z_1, z_2)) = \iota_{S_1}(x_1, y_1, z_1) + \iota_{S_2}(x_2, y_2, z_2) \ .$$

(M4) *If $\Phi : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ is an equivariant holomorphic embedding of bounded symmetric domains of tube type of equal rank, then $\iota_{S_2} \circ \Phi = \iota_{S_1}$.*
(M5) *It is normalized by $\iota_{\mathbb{T}}(1, -1, -i) = 1$ for the Shilov boundary $\mathbb{T}$ of the unit disc Δ.*

5 More on orbits on products of flag manifolds

The Shilov boundary S of a bounded domain is in particular a generalized flag manifold of G, i.e. of the form G/P, where P is a parabolic subgroup of G. A nice description of P is obtained after performing a Cayley transform. The domain $\mathcal{D}$ is transformed to an unbounded domain $\mathcal{D}^C$ which is a Siegel domain of type II and the group P is the group of all affine transformations preserving $\mathcal{D}^C$. The group P has some specific properties: it is a maximal parabolic subgroup of G, conjugate to its opposite. Moreover, one can show that the domain $\mathcal{D}$ is of tube type if and only if the unipotent radical U of P is abelian. A natural question arises to which extent results similar to Theorem 3 could be valid for other generalized flag manifolds. The natural background for this problem is the following. If $P_1, \ldots, P_k$ are parabolic subgroups of a connected semisimple group G', then the product manifold

$$M := G'/P_1 \times \ldots \times G'/P_k$$

is called a multiple flag manifold of finite type if the diagonal action of G' on M has only finitely many orbits. For $k = 1$ we always have only one orbit, and for $k = 2$ the finiteness of the set of orbits follows from the Bruhat decomposition of G'. For $G' = \mathrm{GL}_n(\mathbb{K})$ or $G' = \mathrm{Sp}_{2n}(\mathbb{K})$ and $\mathbb{K}$ an algebraically closed field of characteristic zero, it has been shown in [MWZ99/00] that finite type implies $k \leq 3$, and for $k = 3$ the triples of parabolics leading to multiple flag manifolds of finite type are described and the G'-orbits in these manifolds classified. The main technique to achieve these classifications was the representation theory of quivers. In [Li94], Littelmann considers general simple algebraic groups over $\mathbb{K}$ and describes all multiple flag manifolds of finite type for $k = 3$ under the assumption that P_1 is a Borel subgroup and P_2, P_3 are maximal parabolics. Actually Littelmann considers the condition that $B = P_1$ has a dense orbit in $G'/P_2 \times G'/P_3$, but the results in [Vi86] show that this implies the finiteness of the number of B-orbits and hence the finiteness of the number of G'-orbits in $G'/B \times G'/P_2 \times G'/P_3$. From Littelmann's classification one can easily read off that for a maximal parabolic P in G' the triple product $(G'/P)^3$ is of finite type if and only if the unipotent radical U of P is abelian, and in two exceptional situations. If U is abelian, then P is the maximal parabolic defined by a 3-grading of $\mathfrak{g}' = \mathbf{L}(G')$, so that G'/P is the conformal completion of a Jordan triple (cf. [BN05] for a discussion of such completions in an abstract setting). This case was also studied in [RRS92]. The first exceptional case, where U is not abelian, corresponds to $G' = \mathrm{Sp}_{2n}(\mathbb{K})$, where $G'/P = \P_{2n-1}(\mathbb{K})$ is the projective space of $\mathbb{K}^{2n}$, U is the $(2n-1)$-dimensional Heisenberg group and the Levi complement is $\mathrm{Sp}_{2n-2}(\mathbb{K}) \times \mathbb{K}^\times$. In the other exceptional case $G' = \mathrm{SO}_{2n}(\mathbb{K})$, and G'/P is the highest weight orbit in the 2^n-dimensional spin representation of the covering group $\tilde{G}' = \mathrm{Spin}_{2n}(\mathbb{K})$ of G'. Here $U \cong \Lambda^2(\mathbb{K}^n) \oplus \mathbb{K}^n$ also is a 2-step nilpotent group and the Levi complement acts like $\mathrm{GL}_n(\mathbb{K})$ on this group. A classification of all spaces G/P (G reductive algebraic over an algebraically closed field) for which G has open orbits in $(G/P)^3$ has been obtained recently by V. Popov in [Po05].

It seems that the positive finiteness results have a good chance to carry over to the split forms of groups over more general fields and in particular to $\mathbb{K} = \mathbb{R}$. Suppose that G is a real reductive groups and $G_\mathbb{C}$ its complexification, and that $G_\mathbb{C}$ acts with finitely many orbits on $(G_\mathbb{C}/P_\mathbb{C})^3$. Then, for each $G_\mathbb{C}$-orbit $M \subseteq (G_\mathbb{C}/P_\mathbb{C})^3$ meeting the totally real submanifold $(G/P)^3$, the intersection $M \cap (G/P)^3$ is totally real in M, hence a real form of M, and [BS64, Cor. 6.4] implies that G has only finitely many orbits in $M \cap (G/P)^3$.

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The present note is an extended abstract of a talk given by the second author at the ESI. Complete proofs and details appear in [CN05]. We also refer to [CN05] for more detailed references.

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On a certain 8-dimensional non-symmetric homogenous convex cone

Takaaki Nomura

This work started with a discussion with Simon Gindikin when I visited Rutgers University last year. The idea of considering the present 8-dimensional cone comes from just looking through a list of basic relative invariants associated to homogeneous convex cones given by Yusuke Watanabe who is currently preparing his master thesis. Discussions with Hideyuki Ishi also contribute to the contents.

Let us begin with some facts about matrices. Consider the following subgroups $A_{\mathbb{C}}$ and $N_{\mathbb{C}}$ of $GL(r,\mathbb{C})$:

$$A_{\mathbb{C}} := \big\{ a = \mathrm{diag}[a_1,\dots,a_r] \ ;\ a_1 \in \mathbb{C}^{\times},\dots,a_r \in \mathbb{C}^{\times} \big\},$$

$$N_{\mathbb{C}} := \left\{ n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ n_{21} & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ n_{r-1,1} & n_{r-1,2} & & 1 & 0 \\ n_{r1} & n_{r2} & \cdots & n_{r,r-1} & 1 \end{pmatrix} \ ;\ n_{ji} \in \mathbb{C}\,(j > i) \right\}.$$

We denote by V the vector space of real $r \times r$ symmetric matrices: $V := \mathrm{Sym}(r,\mathbb{R})$. In V, we have the open convex cone Ω of positive definite matrices. Then the tube domain $\Omega + iV$ in the complexified vector space $V_{\mathbb{C}}$ is contained in the orbit of the triangular subgroup $N_{\mathbb{C}}A_{\mathbb{C}}$ through the $r \times r$ unit matrix $E \in \Omega$:

$$\Omega + iV \subset N_{\mathbb{C}}A_{\mathbb{C}} \cdot E. \tag{0.1}$$

For every real or complex $r \times r$ matrix $w = (w_{ij})$, we denote by $\Delta_k(w)$ the k-th principal minor of w $(k = 1,\dots,r)$:

$$\Delta_k(w) = \det \begin{pmatrix} w_{11} & \cdots & w_{1k} \\ \vdots & & \vdots \\ w_{k1} & \cdots & w_{kk} \end{pmatrix},$$

and we set $\Delta_0(w) \equiv 1$. By (0.1) we see that if $w \in \Omega + iV$, then we have $\Delta_k(w) \neq 0$ for any $k = 1,\dots r$.

Lemma 0.1. *Let $w \in \mathrm{Sym}(r,\mathbb{C})$ and suppose that $\mathrm{Re}\,w \in \Omega$. If one writes $w = na\,{}^t\!n$ with $a = \mathrm{diag}[a_1,\dots,a_r] \in A_{\mathbb{C}}$ and $n \in N_C$ according to* (0.1)*, then one has*

$$a_k = \frac{\Delta_k(w)}{\Delta_{k-1}(w)} \qquad (k = 1,\dots,r).$$

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Lemma 0.2. *Suppose that* $\mathrm{Re}(na{}^t\!n)$ *is positive definite for* $a = \mathrm{diag}[a_1, \dots, a_r] \in A_{\mathbb{C}}$ *and* $n \in N_{\mathbb{C}}$. *Then* $\mathrm{Re}\, a_1 > 0, \dots, \mathrm{Re}\, a_r > 0$.

From these two lemmas we have the following proposition.

Proposition 0.3. *Let* $w \in \mathrm{Sym}(r, \mathbb{C})$ *and suppose that* $\mathrm{Re}\, w \in \Omega$. *Then*

$$\mathrm{Re}\, \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \qquad (k = 1, \dots, r).$$

By using the framework of Euclidean Jordan algebra, it is not hard to generalize Proposition 0.3 to the case where Ω is a general symmetric cone. For this we just think of $\Delta_k(w)$ as the Jordan algebra principal minors which are described in the book of Faraut–Korányi [1]. In view of the fact that Lemma 0.2 can be proved by making use of the stability of the Jordan algebra inverse map $w \mapsto w^{-1}$ for symmetric tube domains, and then of the fact that this stability characterizes symmetric tube domains (cf. Kai–Nomura [4]), it would be quite natural to have the following question:

Question 0.4. *Is Proposition 0.3 characteristic of symmetric cones?*

Here if one would like to generalize Proposition 0.3 to any homogenous open convex cone (we say *homogenous cone* in what follows for simplicity), then it is necessary to generalize Δ_k to such a case. Ishi has done this in [3], and we take $\Delta_k(w)$ as the basic relative invariants associated to homogeneous cones following [3]. These are polynomial functions on the ambient vector space V of the homogeneous cone Ω under consideration, so that they are naturally continued to holomorphic polynomial functions on the complexification $V_{\mathbb{C}}$. Of course, if the cone is symmetric, these basic relative invariants coincide with the principal minors. Now Question 0.4 can be formulated in the following way:

Conjecture 0.5. *With the above notation, the implication for* $w \in V_{\mathbb{C}}$ *that*

$$\mathrm{Re}\, w \in \Omega \implies \mathrm{Re}\, \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \dots, r) \tag{$*$}$$

is equivalent to the symmetry of Ω.

The purpose of this note is to present a counterexample to this conjecture. In other words, we show that there is a non-symmetric homogeneous cone for which we have the above implication $(*)$ for elements $w \in V_{\mathbb{C}}$. The cone is 8-dimensional as mentioned in the title.

From now on, let I be the 2×2 unit matrix, and V will denote the 8-dimensional real vector space described in the following manner:

$$V := \left\{ x = \begin{pmatrix} x_{11} I & x_{21} I & \mathbf{y} \\ x_{21} I & x_{22} I & \mathbf{z} \\ {}^t\mathbf{y} & {}^t\mathbf{z} & x_{33} \end{pmatrix} \; ; \; \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2, \; \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^2, \; x_{ij} \in \mathbb{R} \right\}.$$

We note $V \subset \mathrm{Sym}(5, \mathbb{R})$. As for an open convex cone Ω, we take the positive definite ones in V:

$$\Omega := \{x \in V \; ; \; x \gg 0\}. \tag{0.2}$$

Let

$$
\begin{aligned}
&A := \{a = \operatorname{diag}[a_1 I, a_2 I, a_3]\ ;\ a_1 > 0,\ a_2 > 0,\ a_3 > 0\}, \\
&N := \left\{ n = \begin{pmatrix} I & 0 & 0 \\ \xi I & I & 0 \\ {}^t\mathbf{n}_1 & {}^t\mathbf{n}_2 & 1 \end{pmatrix}\ ;\ \xi \in \mathbb{R},\ \mathbf{n}_1 \in \mathbb{R}^2,\ \mathbf{n}_2 \in \mathbb{R}^2 \right\}.
\end{aligned} \tag{0.3}
$$

Then we see without difficulty that the semidirect product group $H := N \ltimes A$ acts on Ω by $H \times \Omega \ni (h, x) \mapsto hx{}^th \in \Omega$ simply transitively. Indeed, if $x \in \Omega$ is expressed as in the definition of V, then the equation $x = na{}^tn$ with $x \in A$ and $n \in N$ as in (0.3) is solved as

$$
\begin{aligned}
a_1 &= \Delta_1(x), & a_2 &= \frac{\Delta_2(x)}{\Delta_1(x)}, & a_3 &= \frac{\Delta_3(x)}{\Delta_2(x)}, \\
\xi &= \frac{x_{21}}{\Delta_1(x)}, & \mathbf{n}_1 &= \frac{\mathbf{y}}{\Delta_1(x)}, & \mathbf{n}_2 &= \frac{x_{11}\mathbf{z} - x_{21}\mathbf{y}}{\Delta_2(x)}.
\end{aligned} \tag{0.4}
$$

Here, $\Delta_1, \Delta_2, \Delta_3$ are the polynomial functions on V given by

$$
\begin{cases}
\Delta_1(x) = x_{11}, \\
\Delta_2(x) = x_{11}x_{22} - x_{21}^2, \\
\Delta_3(x) = x_{11}x_{22}x_{33} + 2x_{21}\mathbf{y}\cdot\mathbf{z} - x_{33}x_{21}^2 - x_{22}\|\mathbf{y}\|^2 - x_{11}\|\mathbf{z}\|^2,
\end{cases}
$$

with $\mathbf{y}\cdot\mathbf{z}$ the canonical inner product in $\mathbb{R}^2$ and $\|\cdot\|$ the corresponding norm. Moreover these $\Delta_k(x)$ $(k = 1, 2, 3)$ are the basic relative invariants associated to the current homogeneous cone Ω. We note here that, if $\delta_k(x)$ $(k = 1, \dots, 5)$ stands for the k-th principal minors of the 5×5 matrix $x \in V$, then

$$
\begin{gathered}
\delta_1(x) = \Delta_1(x), \quad \delta_2(x) = \Delta_1(x)^2, \quad \delta_3(x) = \Delta_1(x)\Delta_2(x), \\
\delta_4(x) = \Delta_2(x)^2, \quad \delta_5(x) = \Delta_2(x)\Delta_3(x).
\end{gathered}
$$

Therefore $x \in \Omega$ is equivalent to $\Delta_k(x) > 0$ for any $k = 1, 2, 3$. Of course this is also seen from the general case treated in Ishi [3].

Now, extending the canonical inner product $\mathbf{y}\cdot\mathbf{z}$ in $\mathbb{R}^2$ to a complex bilinear form on $\mathbb{C}^2$ which we denote by the same symbol, and writing $\nu(\mathbf{y}) := \mathbf{y}\cdot\mathbf{y}$ instead of $\|\mathbf{y}\|^2$ (and similarly for $\nu(\mathbf{z})$), we have the obvious analytic continuations of $\Delta_k(x)$ $(k = 1, 2, 3)$ to holomorphic polynomial functions on $V_{\mathbb{C}}$. Let $A_{\mathbb{C}}$ and $N_{\mathbb{C}}$ be the complexifications of A and N, respectively. As shown in Nomura [6] for the general case, the tube domain $\Omega + iV$ is contained in the orbit of $N_{\mathbb{C}}A_c$ through the 5×5 identity matrix $E \in \Omega$, that is, $\Omega + iV \subset N_{\mathbb{C}}A_{\mathbb{C}} \cdot E$. In particular, none of Δ_k vanishes on $\Omega + iV$.

Proposition 0.6. *Suppose that $w \in V_{\mathbb{C}}$ satisfies $\operatorname{Re} w \in \Omega$. Then*

$$
\operatorname{Re} \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \qquad (k = 1, 2, 3).
$$

Proposition 0.6 is almost clear from the complex version of (0.4) together with Lemma 0.2 applied to the case $r = 5$, our cone (0.2) being evidently a subset of the cone of 5×5 real positive definite symmetric matrices.

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A characterization of symmetric tube domains by convexity of Cayley transform images

Chifune Kai

Abstract

We show that a homogeneous tube domain is symmetric if and only if its Cayley transform image is convex. Moreover this convexity forces the parameter of the Cayley transform to be a specific one, so that the Cayley transform coincides with the standard one defined in terms of the Jordan algebra structure associated with the domain.

1 Introduction

A homogeneous bounded domain is an important geometric and analytic object. It is holomorphically equivalent to a homogeneous Siegel domain, which is a higher dimensional analogue of the right half-plane in $\mathbb{C}$ and is affine homogeneous. Among homogeneous Siegel domains, there is a special subclass consisting of symmetric ones. In [3] we characterized symmetric Siegel domains by the simple geometric condition that the image of the naturally defined Cayley transform is convex. We have this convexity as follows. Since a symmetric Siegel domain is a Hermitian symmetric space of non-compact type, it has a canonical bounded realization, the Harish-Chandra realization. In [6] Korányi and Wolf introduced (the inverses of) the Cayley transforms which map a symmetric Siegel domain to its Harish-Chandra realization. Since the Harish-Chandra realization is known to be the open unit ball for a certain norm, the image of the Cayley transform is convex. Before proceeding, we would like to mention that it is shown in [7] that the Harish-Chandra realization of a symmetric Siegel domain is characterized essentially among bounded realizations by its convexity. In other words, the Cayley transform is essentially the only bounded convex realization of a symmetric Siegel domain.

In this article we deal with homogeneous tube domains (homogeneous Siegel domains of type I) for simplicity. We present an example. We put $V := \mathrm{Sym}(n, \mathbb{R})$, the real vector space of symmetric matrices of order n and set $W := V_{\mathbb{C}} = \mathrm{Sym}(n, \mathbb{C})$. We define an open convex cone Ω by

$$\Omega := \{X \in V \mid X \gg 0 \text{ (positive definite)}\}.$$

Then the tube domain $\Omega + iV \subset W$ is symmetric, which is called the Siegel upper half-plane (though it is a right half-plane here). The Cayley transform for $\Omega + iV$ is introduced by

$$\mathcal{C}(w) := (w - e)(w + e)^{-1} \qquad (w \in W),$$

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where e is the unit matrix of order n. We put $z := \mathcal{C}(w)$. By an easy computation we see that

$$\operatorname{Re} w = (e-z)^{-1}(e-zz^*)\big((e-z)^{-1}\big)^*.$$

Hence we have

$$\mathcal{C}(\Omega + iV) = \{z \in W \mid e - zz^* \gg 0\}.$$

The right hand side is the open unit ball for a certain norm and is convex. For every symmetric tube domain, we can define the Cayley transform in a natural way using the structure of the associated Jordan algebra, and the image of the Cayley transform is convex (see §3 for details). We shall see that this convexity characterizes symmetric tube domains among homogeneous ones.

In this article we deal with the parametrized family of Cayley transforms for homogeneous Siegel domains defined by Nomura [11], which is specialized to tube domains. This family includes Penney's Cayley transform [13], and Nomura's one associated with the Bergman kernel (resp. Szegö kernel) of the domain appearing in [8], [9] and [10] (resp. [12]). Though there is no essential difference between these three Cayley transforms in the case of tube domains, the parametrization is significant when we use the main theorem of this article to prove [3, Theorem 3.1]. If the domain is symmetric, the parametrized family of Cayley transforms includes the above-mentioned Cayley transform defined by means of the Jordan algebra structure. Our theorem states also that the convexity of Cayley transform image is characteristic of that Cayley transform.

2 Homogeneous tube domains

Let V be a finite-dimensional vector space over $\mathbb{R}$. An open convex cone $\Omega \subset V$ is called a *homogeneous convex cone*, if the linear automorphism group

$$G(\Omega) := \{g \in GL(V) \mid g(\Omega) = \Omega\}$$

acts transitively on Ω. We put $W := V_{\mathbb{C}}$, the complexification of V, and denote by $w \mapsto w^*$ the complex conjugation of W with respect to the real form V. For a homogeneous convex cone $\Omega \subset V$, we call the domain $\Omega + iV \subset W$ a *homogeneous tube domain.* It is homogeneous, in particular, affine homogeneous.

2.1 Admissible relative invariants on the cone

By [14, Theorem 1], there exists a split solvable subgroup H of $G(\Omega)$ acting simply transitively on Ω. A function $\Delta : \Omega \to \mathbb{R}_+$ is called a *relative invariant* if there exists a character (one-dimensional representation) of H such that $\Delta(hx) = \chi(h)\Delta(x)$ ($h \in H, x \in \Omega$). We take any $E \in \Omega$ and fix it throughout this article. For a relative invariant Δ on Ω, we define a bilinear form on V by

$$\langle x|y\rangle_\Delta := D_x D_y \log \Delta(E) \qquad (x, y \in V),$$

where for a C^∞ function f on V, $v \in V$ and $x \in \Omega$, we define

$$D_v f(x) := \tfrac{d}{dt} f(x+tv)\big|_{t=0}.$$

If the bilinear form $\langle\cdot|\cdot\rangle_\Delta$ defines a positive definite inner product on V, we say that Δ is *admissible* (only in this article). We know that one of the admissible relative invariants is given by $\Delta_{\mathrm{Det}}(h) := \det h^{-1}$ $(h \in H)$.

3 Cayley transforms for symmetric tube domains

We suppose that $\Omega + iV$ is symmetric in this section. Then Ω is a symmetric cone and the associated Jordan algebra structure is introduced in the following way. We define a commutative product $\circ$ on V by

$$\langle x \circ y|z\rangle_{\Delta_{\mathrm{Det}}} = -\tfrac{1}{2} D_x D_y D_z \log \Delta_{\mathrm{Det}}(E) \qquad (x, y, z \in V). \tag{3.1}$$

We know that E is the unit element. Since $\Omega + iV$ is symmetric, we see that V with the product $\circ$ is a Jordan algebra. This means that in addition to the commutativity we have for all $x, y \in V$,

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y).$$

Moreover this Jordan algebra is Euclidean in the sense of [1]. In fact, by (3.1) we have

$$\langle x \circ y|z\rangle_{\Delta_{\mathrm{Det}}} = \langle x|y \circ z\rangle_{\Delta_{\mathrm{Det}}} \qquad (x, y, z \in V).$$

We extend the product $\circ$ to W by complex bilinearity. Then W is a semisimple complex Jordan algebra.

We introduce the *Cayley transform* $\mathcal{C}_{\mathrm{J}}$ for $\Omega + iV$ by

$$\mathcal{C}_{\mathrm{J}}(w) := (w - E) \circ (w + E)^{-1} \qquad (w \in W).$$

Remark 3. *We note that the above definition is rewritten as*

$$\mathcal{C}_{\mathrm{J}}(w) := E - 2(w + E)^{-1} \qquad (w \in W).$$

For an invertible $v \in V$, the Jordan algebra inverse v^{-1} is characterized as

$$\langle v^{-1}|x\rangle_{\Delta_{\mathrm{Det}}} = -D_x \log \Delta_{\mathrm{Det}}(v) \qquad (x \in V).$$

Let us describe the Cayley transform image $\mathcal{C}_{\mathrm{J}}(\Omega + iV)$. For $w \in W$, we denote by $L(w)$ the multiplication operator by w: $L(w)v := w \circ v$ $(v \in W)$. For $x, y \in W$, we define a complex linear operator $x \Box y$ on W by

$$x \Box y := L(x \circ y) + [L(x), L(y)].$$

For $w \in W$, we set $|w| := \|w \Box w^*\|^{1/2}$, where the operator norm is computed through the norm associated with $\langle\cdot|\cdot\rangle_{\Delta_{\mathrm{Det}}}$. By [1, Proposition X.4.1], we know that $|\cdot|$ is a norm on W, which is called the *spectral norm.*

Proposition 3.1. *We have*

$$\mathcal{C}_{\mathrm{J}}(\Omega + iV) = \{w \in W \mid |w| < 1\},$$

so that $\mathcal{C}_{\mathrm{J}}(\Omega + iV)$ is convex.

4 Cayley transforms for homogeneous tube domains

Now we proceed to the case of homogeneous tube domains. Let $\Omega+iV$ be the homogeneous tube domain defined in §2. Let Δ be any admissible relative invariant on Ω. For $x \in \Omega$, the *pseudoinverse* $\mathcal{I}_\Delta(x)$ of x is defined by

$$\langle \mathcal{I}_\Delta(x) | y \rangle_\Delta = -D_y \log \Delta(x) \qquad (y \in V).$$

We call $\mathcal{I}_\Delta : \Omega \to V$ the *pseudoinverse map.* Let us present the key properties of $\mathcal{I}_\Delta$:

- We denote by Ω^Δ the dual cone of Ω realized in V by means of the inner product $\langle\cdot|\cdot\rangle_\Delta$. We see that $\mathcal{I}_\Delta$ gives a bijection from Ω onto Ω^Δ.
- One has $\mathcal{I}_\Delta(E) = E$.
- $\mathcal{I}_\Delta$ is analytically continued to a birational map $W \to W$. Let $H_{\mathbb{C}}$ be the complexification of H. We extend $\langle\cdot|\cdot\rangle_\Delta$ to a complex bilinear form on W. Then $\mathcal{I}_\Delta$ is $H_{\mathbb{C}}$-equivariant: $\mathcal{I}_\Delta(hx) = {}^\Delta h^{-1}\mathcal{I}_\Delta(x)$ $(h \in H_{\mathbb{C}})$, where ${}^\Delta h$ stands for the transpose of h with respect to $\langle\cdot|\cdot\rangle_\Delta$.
- $\mathcal{I}_\Delta$ is holomorphic on $\Omega + iV$.
- We suppose that $\Omega+iV$ is symmetric and $\Delta = {\Delta_{\mathrm{Det}}}^p$ for some $p > 0$. We introduce the Jordan algebra structure with the unit element E as we did in §3. Then $\mathcal{I}_\Delta$ coincides with the Jordan algebra inverse map.

We define the *Cayley transform* $\mathcal{C}_\Delta$ for $\Omega + iV$ by

$$\mathcal{C}_\Delta(w) := E - 2\mathcal{I}_\Delta(w + E) \qquad (w \in W).$$

It is shown that $\mathcal{C}_\Delta$ maps $\Omega+iV$ biholomorphically onto a bounded domain. Thus we have the Cayley transform for every admissible relative invariant on Ω. Our characterization theorem for symmetric tube domains is stated as follows:

Theorem 4. *Let $\Omega + iV$ be a homogeneous tube domain and Δ an admissible relative invariant on Ω. Then $\mathcal{C}_\Delta(\Omega + iV)$ is convex if and only if $\Omega + iV$ is symmetric and $\Delta = {\Delta_{\mathrm{Det}}}^p$ for some $p > 0$.*

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The asymptotic expansion of Bergman kernels on symplectic manifolds

George Marinescu

1 Introduction

In this talk, we explain some ideas of our approach to the asymptotic expansion of the Bergman kernel associated to a line bundle. The basic philosophy developed in [8, 15, 18] is that the spectral gap properties for the operators proved in [3, 14] implies the existence of the asymptotic expansion for the corresponding Bergman kernels if the manifold X is compact or not, or singular, or with boundary, by using the analytic localization technique inspired by [2, §11]. The interested readers may find complete references in [8, 15, 17], and in the forthcoming book [18].

We consider a compact complex manifold (X, J) with complex structure J, and holomorphic vector bundles L, E on X, with $\operatorname{rk} L = 1$. Let $\{H^{0,q}(X, L^p \otimes E)\}_{q=0}^n$ be the Dolbeault cohomology groups of the Dolbeault complex $(\Omega^{0,\bullet}(X, L^p \otimes E), \overline{\partial}^{L^p\otimes E}) := (\oplus_q \Omega^{0,q}(X, L^p \otimes E), \overline{\partial}^{L^p\otimes E})$.

We fix Hermitian metrics h^L, h^E on L, E. Let ∇^L be the holomorphic Hermitian connection on (L, h^L) with curvature R^L and let g^{TX} be a Riemannian metric on X such that

$$g^{TX}(J\cdot, J\cdot) = g^{TX}(\cdot, \cdot). \tag{1.1}$$

We denote by $\overline{\partial}^{L^p\otimes E,*}$ the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ on the Dolbeault complex $\Omega^{0,\bullet}(X, L^p \otimes E)$ endowed with the L^2-scalar product associated to the metrics h^L, h^E and g^{TX} and the Riemannian volume form $dv_X(x')$. Set

$$D_p = \sqrt{2}\big(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}\big). \tag{1.2}$$

Then $\frac{1}{2}D_p^2$ is the Kodaira-Laplacian acting on $\Omega^{0,\bullet}(X, L^p \otimes E)$ and preserves its $\mathbb{Z}$-grading. By Hodge theory, we know that

$$\operatorname{Ker} D_p|_{\Omega^{0,q}} = \operatorname{Ker} D_p^2|_{\Omega^{0,q}} \simeq H^{0,q}(X, L^p \otimes E). \tag{1.3}$$

We denote by P_p the orthogonal projection from $\Omega^{0,\bullet}(X, L^p \otimes E)$ onto $\operatorname{Ker} D_p$. The Bergman kernel $P_p(x, x')$, $(x, x' \in X)$ of $L^p \otimes E$ is the smooth kernel of P_p with respect to the Riemannian volume form $dv_X(x')$.

In this setting, we are interested to understand the asymptotic expansion of $P_p(x, x')$ as $p \to \infty$. If R^L is positive, it is studied in [22, 20, 26, 7, 4, 21, 13, 23, 12] in various generalities. Moreover, the coefficients in the diagonal asymptotic expansion encode geometric information about the underlying complex projective manifolds. This diagonal asymptotic expansion plays a crucial role in the recent work of Donaldson [10] where the existence of Kähler metrics with constant scalar curvature is shown to be closely related to Chow-Mumford stability.

In the symplectic setting, Dai, Liu and Ma [8] studied the asymptotic expansion of the Bergman kernel of the spin^c Dirac operator associated to a positive line bundle on compact symplectic manifolds, and related it to that of the corresponding heat kernel. This approach is inspired by local Index Theory, especially by the analytic localization techniques of Bismut-Lebeau [2, §11]. In [8] they also focused on the full off-diagonal asymptotic expansion [8, Theorem 4.18] which is needed to study the Bergman kernel on orbifolds. By exhibiting the spectral gap properties of the corresponding operators, in [17], we also explained that without changing any step in the proof of [8, Theorem 4.18], the full off-diagonal asymptotic expansion still holds in the complex or symplectic setting if the curvature R^L of L is only non-degenerate. Note that Berman and Sjöstrand [1] recently also studied the asymptotic expansion in the complex setting when the curvauture R^L of L is only non-degenerate.

Along with $\frac{1}{2}D_p^2$ there is another geometrically defined generalization to symplectic manifolds of the Kodaira-Laplace operator, namely the renormalized Bochner-Laplacian. In this talk, we explain the asymptotic expansion of the generalized Bergman kernels of the renormalized Bochner-Laplacian on high tensor powers of a positive line bundle on compact symplectic manifolds. In this situation the operators have small eigenvalues when the power $p \to \infty$ (the only small eigenvalue is zero in [8], thus we have the key equation [8, (3.89)]) and we are interested in obtaining Theorem 8, that is, the *near* diagonal expansion of the generalized Bergman kernels.

There are three steps: In Step 1, by using the finite propagation speed of solutions of hyperbolic equations, we can localize our problem if the spectral gap properties holds. In Step 2, we work on $T_{x_0}X \simeq \mathbb{R}^{2n}$, and extend the bundles and connections from a neighborhood of 0 to all of $T_{x_0}X$ such that the curvature of the line bundle L is uniformly non-degenerate on $T_{x_0}X$. We combine the Sobolev norm estimates from [8] and a formal power series method to obtain the asymptotic expansion. In Step 3 we compute the coefficients by using the formal power series method from Step 2. Actually, in [17], we compute also some coefficients in the asymptotic expansion of the Bergman kernel associated to the spin^c Dirac operators in [8] by using the formal power series method here.

2 Main results

Let (X, ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}R^L = \omega$, where $R^L = (\nabla^L)^2$ is the curvature of (L, ∇^L). Let (E, h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and its curvature R^E.

Let g^{TX} be a Riemannian metric on X. Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}) with its curvature R^{TX} and its scalar curvature r^X. Let dv_X be the Riemannian volume form of (TX, g^{TX}). The scalar product on the space $\mathscr{C}^\infty(X, L^p \otimes E)$ of smooth sections of $L^p \otimes E$ is given by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle_{L^p \otimes E} \, dv_X(x) \, .$$

Let $\mathbf{J} : TX \longrightarrow TX$ be the skew–adjoint linear map which satisfies the relation

$$\omega(u,v) = g^{TX}(\mathbf{J}u, v) \tag{2.1}$$

for $u, v \in TX$. Let J be an almost complex structure which is (separately) compatible with g^{TX} and ω, especially, $\omega(\cdot, J\cdot)$ defines a metric on TX. Then J commutes also with $\mathbf{J}$. Let $\nabla^X J \in T^*X \otimes \mathrm{End}(TX)$ be the covariant derivative of J induced by ∇^{TX}. Let $\nabla^{L^p\otimes E}$ be the connection on $L^p \otimes E$ induced by ∇^L and ∇^E. Let $\{e_i\}_i$ be an orthonormal frame of (TX, g^{TX}). Set $|\nabla^X J|^2 = \sum_{ij} |(\nabla^X_{e_i} J)e_j|^2$. Let $\Delta^{L^p\otimes E} = -\sum_i [(\nabla^{L^p\otimes E}_{e_i})^2 - \nabla^{L^p\otimes E}_{\nabla^{TX}_{e_i} e_i}]$ be the induced Bochner-Laplacian acting on $\mathscr{C}^\infty(X, L^p \otimes E)$. We fix a smooth hermitian section Φ of $\mathrm{End}(E)$ on X. Set $\tau(x) = -\pi \operatorname{Tr}_{|TX}[J\mathbf{J}]$, and

$$\Delta_{p,\Phi} = \Delta^{L^p\otimes E} - p\tau + \Phi. \tag{2.2}$$

By [14, Cor. 1.2] (cf. also [11, Theorem 2]) there exist μ_0, $C_L > 0$ independent of p such that the spectrum of $\Delta_{p,\Phi}$ satisfies

$$\operatorname{Spec} \Delta_{p,\Phi} \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty[\,. \tag{2.3}$$

This is the spectral gap property which plays an essential role in our approach. In the first place, it indicates a natural space of sections which replace the space of holomorphic sections from the complex case.

Let $P_{0,p}$ be the orthogonal projection from $(\mathscr{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle)$ onto the eigenspace of $\Delta_{p,\Phi}$ with eigenvalues in $[-C_L, C_L]$. If the complex case (i.e. J is integrable and $\Phi = -\frac{\sqrt{-1}}{2} R^E(e_j, Je_j)$) the interval $[-C_L, C_L]$ contains for p large enough only the eigenvalue 0 whose eigenspace consists of holomorphic sections. For the computation of the spectral density function we need more general kernels. Namely, we define $P_{q,p}(x, x')$, $q \geqslant 0$ as the smooth kernels of the operators $P_{q,p} = (\Delta_{p,\Phi})^q P_{0,p}$ (we set $(\Delta_{p,\Phi})^0 = 1$) with respect to $dv_X(x')$. They are called the generalized Bergman kernels of the renormalized Bochner-Laplacian $\Delta_{p,\Phi}$. Let $\det \mathbf{J}$ be the determinant function of $\mathbf{J}_x \in \mathrm{End}(T_x X)$.

Theorem 5. *There exist smooth coefficients $b_{q,r}(x) \in \mathrm{End}(E)_x$ which are polynomials in R^{TX}, R^E (and R^L, Φ) and their derivatives of order $\leqslant 2(r+q)-1$ (resp. $2(r+q)$), and reciprocals of linear combinations of eigenvalues of $\mathbf{J}$ at x, and*

$$b_{0,0} = (\det \mathbf{J})^{1/2} \operatorname{Id}_E, \tag{2.4}$$

such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $x \in X$, $p \in \mathbb{N}$,

$$\Big| \frac{1}{p^n} P_{q,p}(x,x) - \sum_{r=0}^{k} b_{q,r}(x) p^{-r} \Big|_{\mathscr{C}^l} \leqslant C_{k,l}\, p^{-k-1}. \tag{2.5}$$

Moreover, the expansion is uniform in that for any $k, l \in \mathbb{N}$, there is an integer s such that if all data (g^{TX}, h^L, ∇^L, h^E, ∇^E, J and Φ) run over a bounded set in the $\mathscr{C}^s$-norm and g^{TX} stays bounded below, the constant $C_{k,l}$ is independent of g^{TX}; and the $\mathscr{C}^l$-norm in (2.5) *includes also the derivatives on the parameters.*

Theorem 6. *If $J = \mathbf{J}$, then for $q \geqslant 1$,*

$$b_{0,1} = \frac{1}{8\pi}\Big[r^X + \frac{1}{4}|\nabla^X J|^2 + 2\sqrt{-1} R^E(e_j, Je_j) \Big], \tag{2.6}$$

$$b_{q,0} = \Big(\frac{1}{24}|\nabla^X J|^2 + \frac{\sqrt{-1}}{2} R^E(e_j, Je_j) + \Phi \Big)^q. \tag{2.7}$$

Theorem 5 for $q = 0$ and (2.6) generalize the results of [7], [26], [13] and [23] to the symplectic case. The term $r^X + \frac{1}{4}|\nabla^X J|^2$ in (2.6) is called the Hermitian scalar curvature in the literature and is a natural substitute for the Riemannian scalar curvature in the almost-Kähler case. It was used by Donaldson [9] to define the moment map on the space of compatible almost-complex structures. We can view (2.7) as an extension and refinement of the results of [11, §5] about the density of states function of $\Delta_{p,\Phi}$, (2.7) implies also a correction of a formula in [6].

Now, we try to explain the near-diagonal expansion of $P_{q,p}(x,x')$.

Let a^X be the injectivity radius of (X, g^{TX}). We fix $\varepsilon \in]0, a^X/4[$. We denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in X and T_xX with center x and radius ε. We identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ by using the exponential map of (X, g^{TX}).

We fix $x_0 \in X$. For $Z \in B^{T_{x_0}X}(0,\varepsilon)$ we identify L_Z, E_Z and $(L^p \otimes E)_Z$ to L_{x_0}, E_{x_0} and $(L^p \otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p\otimes E}$ along the curve $\gamma_Z : [0,1] \ni u \to \exp^X_{x_0}(uZ)$. Then under our identification, $P_{q,p}(Z,Z')$ is a section of $\mathrm{End}(E)_{x_0}$ on $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \le \varepsilon$, we denote it by $P_{q,p,x_0}(Z,Z')$. Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX on X. Then we can view $P_{q,p,x_0}(Z,Z')$ as a smooth section of $\pi^* \mathrm{End}(E)$ on $TX \times_X TX$ (which is defined for $|Z|, |Z'| \le \varepsilon$) by identifying a section $S \in \mathscr{C}^\infty(TX \times_X TX, \pi^* \mathrm{End}(E))$ with the family $(S_x)_{x\in X}$. We denote by $|\ \ |_{\mathscr{C}^s(X)}$ a $\mathscr{C}^s$ norm on it for the parameter $x_0 \in X$.

We will define the function $P^N(Z,Z')$ in (4.5).

Theorem 7. *There exist $J_{q,r}(Z,Z') \in \mathrm{End}(E)_{x_0}$ polynomials in Z, Z' with the same parity as r and $\deg J_{q,r}(Z,Z') \le 3r$, whose coefficients are polynomials in R^{TX}, R^E (and R^L, Φ) and their derivatives of order $\leqslant r-1$ (resp. r), and reciprocals of linear combinations of eigenvalues of $\mathbf{J}$ at x_0, such that if we denote by*

$$\mathscr{F}_{q,r}(Z,Z') = J_{q,r}(Z,Z')P^N(Z,Z'), \tag{2.8}$$

then for $k, m, m' \in \mathbb{N}$, $\sigma > 0$, there exists $C > 0$ such that if $t \in]0,1]$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma/\sqrt{p}$,

$$\sup_{|\alpha|+|\alpha'|\leqslant m} \Big| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Big(P_{q,p}(Z,Z') - \sum_{r=2q}^{k} \mathscr{F}_{q,r}(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} \Big) \Big|_{\mathscr{C}^{m'}(X)} \leqslant Cp^{-(k-m+1)/2}. \tag{2.9}$$

3 Idea of the proofs

3.1 Localization

First, (2.3) and the finite propagation speed for hyperbolic equations, allows us to localize the problem. In particular, the asymptotics of $P_{q,p}(x_0,x')$ as $p \to \infty$ are localized on a neighborhood of x_0. Thus we can translate our analysis from X to the manifold $\mathbb{R}^{2n} \simeq T_{x_0}X =: X_0$.

Let $f : \mathbb{R} \to [0,1]$ be a smooth even function such that $f(v) = 1$ for $|v| \leqslant \varepsilon/2$, and $f(v) = 0$ for $|v| \geqslant \varepsilon$. Set

$$F(a) = \Big(\int_{-\infty}^{+\infty} f(v)dv \Big)^{-1} \int_{-\infty}^{+\infty} e^{iva} f(v)dv. \tag{3.1}$$

Then $F(a)$ is an even function and lies in the Schwartz space $\mathcal{S}(\mathbb{R})$ and $F(0) = 1$. Let $\widetilde{F}$ be the holomorphic function on $\mathbb{C}$ such that $\widetilde{F}(a^2) = F(a)$. The restriction of $\widetilde{F}$ to $\mathbb{R}$ lies in the Schwartz space $\mathcal{S}(\mathbb{R})$. Then there exists $\{c_j\}_{j=1}^{\infty}$ such that for any $k \in \mathbb{N}$, the function

$$F_k(a) = \widetilde{F}(a) - \sum_{j=1}^{k} c_j a^j \widetilde{F}(a), \tag{3.2}$$

verifies

$$F_k^{(i)}(0) = 0 \quad \text{for any } 0 < i \leqslant k. \tag{3.3}$$

Proposition 3.1. *For any $k, m \in \mathbb{N}$, there exists $C_{k,m} > 0$ such that for $p \geqslant 1$*

$$\Big| F_k\big(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\big)(x,x') - P_{0,p}(x,x') \Big|_{\mathscr{C}^m(X\times X)} \leqslant C_{k,m} p^{-\frac{k}{2}+2(2m+2n+1)}. \tag{3.4}$$

Here the $\mathscr{C}^m$ norm is induced by ∇^L, ∇^E, h^L, h^E and g^{TX}.

Using (3.1), (3.2) and the finite propagation speed of solutions of hyperbolic equations, it is clear that for $x, x' \in X$, $F_k\big(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\big)(x,\cdot)$ only depends on the restriction of $\Delta_{p,\Phi}$ to $B^X(x, \varepsilon p^{-\frac{1}{4}})$, and $F_k\big(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\big)(x,x') = 0$, if $d(x,x') \geqslant \varepsilon p^{-\frac{1}{4}}$. This means that the asymptotic of $\Delta^q_{p,\Phi} P_{\mathcal{H}_p}(x,\cdot)$ when $p \to +\infty$, modulo $\mathscr{O}(p^{-\infty})$ (i.e. terms whose $\mathscr{C}^m$ norm is $\mathscr{O}(p^{-l})$ for any $l, m \in \mathbb{N}$), only depends on the restriction of $\Delta_{p,\Phi}$ to $B^X(x, \varepsilon p^{-\frac{1}{4}})$.

3.2 Uniform estimate of the generalized Bergman kernels

We will work on the normal coordinate for $x_0 \in X$. We identify the fibers of (L, h^L), (E, h^E) with $(L_{x_0}, h^{L_{x_0}})$, $(E_{x_0}, h^{E_{x_0}})$ respectively, in a neighborhood of x_0, by using the parallel transport with respect to ∇^L, ∇^E along the radial direction.

We then extend the bundles and connections from a neighborhood of 0 to all of $T_{x_0}X$. In particular, we can extend ∇^L (resp. ∇^E) to a Hermitian connection ∇^{L_0} on $(L_0, h^{L_0}) = (X_0 \times L_{x_0}, h^{L_{x_0}})$ (resp. ∇^{E_0} on $(E_0, h^{E_0}) = (X_0 \times E_{x_0}, h^{E_{x_0}})$) on $T_{x_0}X$ in such a way so that we still have positive curvature R^{L_0}; in addition $R^{L_0} = R^L_{x_0}$ outside a compact set. We also extend the metric g^{TX_0}, the almost complex structure J_0, and the smooth section Φ_0, (resp. the connection ∇^{E_0}) in such a way that they coincide with their values at 0 (resp. the trivial connection) outside a compact set. Moreover, using a fixed unit vector $S_L \in L_{x_0}$ and the above discussion, we construct an isometry $E_0 \otimes L_0^p \simeq E_{x_0}$. Let $\Delta^{X_0}_{p,\Phi_0}$ be the renormalized Bochner-Laplacian on X_0 associated to the above data by a formula analogous to (2.2). Then (2.3) still holds for $\Delta^{X_0}_{p,\Phi_0}$ with μ_0 replaced by $4\mu_0/5$.

Let dv_{TX} be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$ and $\kappa(Z)$ be the smooth positive function defined by the equation $dv_{X_0}(Z) = \kappa(Z) dv_{TX}(Z)$, with $k(0) = 1$. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, E_{x_0})$, $Z \in \mathbb{R}^{2n}$ and $t = 1/\sqrt{p}$, set

$$\|s\|_0^2 = \int_{\mathbb{R}^{2n}} |s(Z)|^2_{h^{E_{x_0}}} dv_{TX}(Z),$$

and consider

$$\mathscr{L}_t = S_t^{-1} t^2 \kappa^{\frac{1}{2}} \Delta^{X_0}_{p,\Phi_0} \kappa^{-\frac{1}{2}} S_t\,, \quad \text{where } (S_t s)(Z) = s(Z/t)\,. \tag{3.5}$$

Then $\mathscr{L}_t$ is a family of self-adjoint differential operators with coefficients in $\mathrm{End}(E)_{x_0}$. We denote by $\mathcal{P}_{0,t} : (\mathscr{C}^\infty(X_0, E_{x_0}), \|\ \ \|_0) \to (\mathscr{C}^\infty(X_0, E_{x_0}), \|\ \ \|_0)$ the spectral projection of $\mathscr{L}_t$ corresponding to the interval $[-C_{L_0}t^2, C_{L_0}t^2]$. Let $\mathcal{P}_{q,t}(Z, Z') = \mathcal{P}_{q,t,x_0}(Z, Z')$, $(Z, Z' \in X_0,\ q \geqslant 0)$ be the smooth kernel of $\mathcal{P}_{q,t} = (\mathscr{L}_t)^q\, \mathcal{P}_{0,t}$ with respect to $dv_{TX}(Z')$. We can view $\mathcal{P}_{q,t,x}(Z, Z')$ as a smooth section of $\pi^* \mathrm{End}(E)$ over $TX \times_X TX$, where $\pi : TX \times_X TX \to X$. Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\mu_0/4$. By (2.3),

$$\mathcal{P}_{q,t} = \frac{1}{2\pi i} \int_\delta \lambda^q (\lambda - \mathscr{L}_t)^{-1} d\lambda. \tag{3.6}$$

From (2.3) and (3.6) we can apply the techniques in [8], which are inspired by [2, §11], to get the following key estimate.

Theorem 8. *There exist smooth sections $F_{q,r} \in \mathscr{C}^\infty(TX \times_X TX, \pi^* \mathrm{End}(E))$ such that for $k, m, m' \in \mathbb{N}$, $\sigma > 0$, there exists $C > 0$ such that if $t \in]0, 1]$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma$,*

$$\sup_{|\alpha|,|\alpha'| \leqslant m} \Big| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Big(\mathcal{P}_{q,t} - \sum_{r=0}^k F_{q,r} t^r \Big)(Z, Z') \Big|_{\mathscr{C}^{m'}(X)} \leqslant C t^k. \tag{3.7}$$

Let $P_{0,q,p}(Z, Z') \in \mathrm{End}(E_{x_0})$ $(Z, Z' \in X_0)$ be the analogue of $P_{q,p}(x, x')$. By (3.5), for $Z, Z' \in \mathbb{R}^{2n}$,

$$P_{0,\,q,\,p}(Z, Z') = t^{-2n-2q}\, \kappa^{-\frac{1}{2}}(Z)\, \mathcal{P}_{q,t}(Z/t, Z'/t)\, \kappa^{-\frac{1}{2}}(Z'). \tag{3.8}$$

By Proposition 3.1, we know that

$$P_{0,q,p}(Z, Z') = P_{q,p,x_0}(Z, Z') + \mathscr{O}(p^{-\infty}), \tag{3.9}$$

uniformly for $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leq \varepsilon/2$.

To complete the proof the Theorem 2.1, we finally prove $F_{q,r} = 0$ for $r < 2q$. In fact, (3.7) and (3.8) yield

$$b_{q,r}(x_0) = F_{q,2r+2q}(0, 0). \tag{3.10}$$

4 Evaluation of $F_{q,r}$

The almost complex structure J induces a splitting $T_\mathbb{R}X \otimes_\mathbb{R} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. We choose $\{w_i\}_{i=1}^n$ to be an orthonormal basis of $T^{(1,0)}_{x_0}X$, such that

$$-2\pi\sqrt{-1}\mathbf{J}_{x_0} = \mathrm{diag}(a_1, \cdots, a_n) \in \mathrm{End}(T^{(1,0)}_{x_0}X). \tag{4.1}$$

We use the orthonormal basis $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j = 1, \ldots, n$ of $T_{x_0}X$ to introduce the normal coordinates as in Section 3. In what follows we will use the complex coordinates $z = (z_1, \cdots, z_n)$, thus $Z = z + \overline{z}$, and $w_i = \sqrt{2}\frac{\partial}{\partial z_i}$, $\overline{w}_i = \sqrt{2}\frac{\partial}{\partial \overline{z}_i}$. It is very useful to introduce the creation and annihilation operators b_i, b_i^+,

$$b_i = -2\tfrac{\partial}{\partial z_i} + \frac{1}{2} a_i \overline{z}_i\,, \quad b_i^+ = 2\tfrac{\partial}{\partial \overline{z}_i} + \frac{1}{2} a_i z_i\,, \quad b = (b_1, \cdots, b_n)\,. \tag{4.2}$$

Now there are second order differential operators $\mathcal{O}_r$ whose coefficients are polynomials in Z with coefficients being polynomials in R^{TX}, $R^{\det}$, R^E, R^L and their derivatives at x_0, such that

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{\infty} \mathcal{O}_r t^r, \quad \text{with } \mathscr{L}_0 = \sum_i b_i b_i^+ . \tag{4.3}$$

Theorem 9. *The spectrum of the restriction of $\mathscr{L}_0$ to $L^2(\mathbb{R}^{2n})$ is given by*

$$\Big\{2\sum_{i=1}^{n} \alpha_i a_i \,:\, \alpha_i \in \mathbb{N}\Big\}$$

and an orthogonal basis of the eigenspace of $2\sum_{i=1}^{n} \alpha_i a_i$ is given by

$$b^{\alpha}\big(z^{\beta} \exp\big(-\frac{1}{4}\sum_i a_i |z_i|^2\big)\big), \quad \textit{with } \beta \in \mathbb{N}^n . \tag{4.4}$$

Let $N^{\perp}$ be the orthogonal space of $N = \operatorname{Ker} \mathscr{L}_0$ in $(L^2(\mathbb{R}^{2n}, E_{x_0}), \|\ \ \|_0)$. Let P^N, $P^{N^{\perp}}$ be the orthogonal projections from $L^2(\mathbb{R}^{2n}, E_{x_0})$ onto N, $N^{\perp}$, respectively. Let $P^N(Z, Z')$ be the smooth kernel of the operator P^N with respect to $dv_{TX}(Z')$. From (4.4), we get

$$P^N(Z, Z') = \frac{1}{(2\pi)^n} \prod_{i=1}^{n} a_i \ \exp\Big(-\frac{1}{4}\sum_i a_i\big(|z_i|^2 + |z_i'|^2 - 2 z_i \overline{z}_i'\big)\Big). \tag{4.5}$$

Now for $\lambda \in \delta$, we solve for the following formal power series on t, with $g_r(\lambda) \in \operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0}), N)$, $f_r^{\perp}(\lambda) \in \operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0}), N^{\perp})$,

$$(\lambda - \mathscr{L}_t) \sum_{r=0}^{\infty} \Big(g_r(\lambda) + f_r^{\perp}(\lambda)\Big) t^r = \operatorname{Id}_{L^2(\mathbb{R}^{2n}, E_{x_0})} . \tag{4.6}$$

From (3.6), (4.6), we claim that

$$F_{q,r} = \frac{1}{2\pi i}\int_{\delta} \lambda^q g_r(\lambda) d\lambda + \frac{1}{2\pi i}\int_{\delta} \lambda^q f_r^{\perp}(\lambda) d\lambda. \tag{4.7}$$

From Theorem 9, (4.7), the key observation that $P^N \mathcal{O}_1 P^N = 0$, and the residue formula, we can get $F_{q,r}$ by using the operators $\mathscr{L}_0^{-1}$, P^N, $P^{N^{\perp}}$, $\mathcal{O}_i$, $(i \leqslant r)$. This gives us a method to compute $b_{q,r}$ in view of Theorem 9 and (3.10). Especially, for $q > 0, r < 2q$,

$$\begin{aligned}
F_{0,0} &= P^N, \quad F_{q,r} = 0, \qquad\qquad (4.8)\\
F_{q,2q} &= (P^N \mathcal{O}_2 P^N - P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^{\perp}} \mathcal{O}_1 P^N)^q P^N,\\
F_{0,2} &= \mathscr{L}_0^{-1} P^{N^{\perp}} \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^{\perp}} \mathcal{O}_1 P^N - \mathscr{L}_0^{-1} P^{N^{\perp}} \mathcal{O}_2 P^N\\
&\quad + P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^{\perp}} \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^{\perp}} - P^N \mathcal{O}_2 \mathscr{L}_0^{-1} P^{N^{\perp}}\\
&\quad + P^{N^{\perp}} \mathscr{L}_0^{-1} \mathcal{O}_1 P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^{\perp}} - P^N \mathcal{O}_1 \mathscr{L}_0^{-2} P^{N^{\perp}} \mathcal{O}_1 P^N.
\end{aligned}$$

In fact $\mathscr{L}_0$ and $\mathcal{O}_r$ are formal adjoints with respect to $\|\ \ \|_0$; thus in $F_{0,2}$ we only need to compute the first two terms, as the last two terms are their adjoints. This simplifies the computation in Theorem 6.

5 Generalizations to non-compact manifolds

In this section we come back to the case of complex manifolds, which was briefly discussed in the introduction, but focus on non-compact manifolds. Let (X,Θ) be a Hermitian manifold of dimension n, where Θ is the $(1,1)$ form associated to a hermitian metric on X. Given a Hermitian holomorphic bundles L and E on X with $\operatorname{rk} L=1$, we consider the space of L^2 holomorphic sections $H^0_{(2)}(X,L^p\otimes E)$. Let P_p be the orthogonal projection from the space $\boldsymbol{L}^2(X,L^p\otimes E)$ of L^2 sections of $L^p\otimes E$ onto $H^0_{(2)}(X,L^p\otimes E)$. By generalizing the definition from Section 1, we define the Bergman kernel $P_p(x,x')$, $(x,x'\in X)$ to be the Schwartz kernel of P_p with respect to the Riemannian volume form $dv_X(x')$ associated to (X,Θ). By the ellipticity of the Kodaira-Laplacian and Schwartz kernel theorem, we know $P_p(x,x')$ is $\mathscr{C}^\infty$. Choose an orthonormal basis $(S^p_i)_{i=1}^{d_p}$ $(d_p\in\mathbb{N}\cup\{\infty\})$ of $H^0_{(2)}(X,L^p\otimes E)$. The Bergman kernel can then be expressed as

$$P_p(x,x')=\sum_{i=1}^{d_p}S^p_i(x)\otimes(S^p_i(x'))^*\in(L^p\otimes E)_x\otimes(L^p\otimes E)^*_{x'}.$$

Let $K_X=\det(T^{*(1,0)}X)$ be the canonical line bundle of X and $R^{\det}$ be the curvature of K_X^* relative to the metric induced by Θ. The line bundle L is supposed to be positive and we set $\omega=\frac{\sqrt{-1}}{2\pi}R^L$.

We denote by g^{TX}_ω the Riemannian metric associated to ω and by r^X_ω the scalar curvature of g^{TX}_ω. Moreover, let $\alpha_1,\dots,\alpha_n$ be the eigenvalues of ω with respect to Θ $(\alpha_j=a_j/(2\pi),\ j=1,\dots,n$ where $a_1,\dots,a_n$ are defined by (4.1) and (2.1) with g^{TX} the Riemannian metric associated to $\Theta)$. The torsion of Θ is $T=[i(\Theta),\partial\Theta]$, where $i(\Theta)=(\Theta\wedge\cdot)^*$ is the interior multiplication with Θ.

Theorem 10 ([15]). *Assume that (X,Θ) is a complete Hermitian manifold of dimension n. Suppose that there exist $\varepsilon>0$, $C>0$ such that*

$$\sqrt{-1}R^L\geqslant\varepsilon\Theta\,,\quad\sqrt{-1}R^{\det}\geqslant-C\Theta\,,\quad\sqrt{-1}R^E\geqslant-C\Theta\,,\qquad|T|\leqslant C\Theta\,.\tag{5.1}$$

Then the kernel $P_p(x,x')$ has a full off–diagonal asymptotic expansion uniformly on compact sets of $X\times X$ and $P_p(x,x)$ has an asymptotic expansion analogous to (2.5) *uniformly on compact sets of X. Moreover, $b_0=\alpha_1\cdots\alpha_n\operatorname{Id}_E$ and*

$$b_1=\frac{\alpha_1\cdots\alpha_n}{8\pi}\Big[r^X_\omega\operatorname{Id}_E-2\Delta_\omega\Big(\log(\alpha_1\cdots\alpha_n)\Big)\operatorname{Id}_E+4\sum_{j=1}^nR^E(w_{\omega,j},\overline{w}_{\omega,j})\Big]\,,$$

where $\{w_{\omega,j}\}$ is an orthonormal basis of $(T^{(1,0)}X,g^{TX}_\omega)$.

By *full off-diagonal* expansion we mean an expansion analogous to (2.9) where we allow $|Z|,|Z'|\leqslant\sigma$.

Let us remark that if $L=K_X$, the first two conditions in (5.1) are to be replaced by

$$h^L\text{ is induced by }\Theta\text{ and }\sqrt{-1}R^{\det}<-\varepsilon\Theta.\tag{5.2}$$

Moreover, if (X,Θ) is Kähler, the condition on the torsion T is trivially satisfied.

The proof is based on the observation that the Kodaira-Laplacian $\mathbf{2}_p = \frac{1}{2}D_p^2$ acting on $\boldsymbol{L}^2(X, L^p \otimes E)$ has a spectral gap as in (2.3). The proof of Theorem 5 applies then and delivers the result.

Theorem 10 has several applications e.g. holomorphic Morse inequalities on non-compact manifolds (as the well-known results of Nadel-Tsuji [19], see also [15, 25]) or Berezin-Toeplitz quantization (see [18] or the fothcommimg [16]).

We will emphazise in the sequel the Bergman kernel for a singular metric. Let X be a compact complex manifold. A *singular Kähler metric* on X is a closed, strictly positive $(1,1)$-current ω. If the cohomology class of ω in $H^2(X,\mathbb{R})$ is integral, there exists a holomorphic line bundle (L, h^L), endowed with a singular Hermitian metric, such that $\frac{\sqrt{-1}}{2\pi}R^L = \omega$ in the sense of currents. We call (L, h^L) a *singular polarization* of ω.

If we change the metric h^L, the curvature of the new metric will be in the same cohomology class as ω. In this case we speak of a polarization of $[\omega] \in H^2(X,\mathbb{R})$. Our purpose is to define an appropriate notion of polarized section of L^p, possibly by changing the metric of L, and study the associated Bergman kernel.

Corollary 11. *Let (X, ω) be a compact complex manifold with a singular Kähler metric with integral cohomology class. Let (L, h^L) be a singular polarization of $[\omega]$ with strictly positive curvature current having singular support along a proper analytic set Σ. Then the Bergman kernel of the space of polarized sections*

$$H^0_{(2)}(X \smallsetminus \Sigma, L^p) = \left\{u \in L_2^{0,0}(X \smallsetminus \Sigma, L^p\,,\, \Theta_P\,, h^L_\varepsilon) : \overline{\partial}^{L^p} u = 0\right\}$$

has the asymptotic expansion as in Theorem 10 for $X \smallsetminus \Sigma$, where Θ_P is a generalized Poincaré metric on $X \smallsetminus \Sigma$ and h^L_ε is a modified Hermitian metric on L.

Using an idea of Takayama [24], Corollary 11 gives a proof of the Shiffman-Ji-Bonavero-Takayama criterion, about the characterization of Moishezon manifolds by $(1,1)$ positive currents.

We mention further the Berezin-Toeplitz quantization. Assume that X is a complex manifold and let $\mathscr{C}^\infty_{const}(X)$ denote the algebra of smooth functions of X which are constant outside a compact set. For any $f \in \mathscr{C}^\infty_{const}(X)$ we denote for simplicity the operator of multiplication with f still by f and consider the linear operator

$$T_{f,p} : \boldsymbol{L}^2(X, L^p) \longrightarrow \boldsymbol{L}^2(X, L^p)\,, \quad T_{f,p} = P_p\, f\, P_p\,. \tag{5.3}$$

The family $(T_{f,p})_{p\geqslant 1}$ is called a Toeplitz operator. The following result generalizes [5] to non-compact manifolds.

Corollary 12. *We assume that (X, Θ) and (L, h^L) satisfy the same hypothesis as in Theorem 10 or* (5.2)*. Let $f, g \in \mathscr{C}^\infty_{const}(X)$. The product of the two corresponding Toeplitz operators admits the asymptotic expansion*

$$T_{f,p}T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + \mathscr{O}(p^{-\infty}) \tag{5.4}$$

where C_r are differential operators. More precisely,

$$C_0(f,g) = fg\,, \quad C_1(f,g) - C_1(g,f) = \frac{1}{\sqrt{-1}}\{f,g\} \tag{5.5}$$

where the Poisson bracket is taken with respect to the metric $2\pi\omega$. Therefore

$$[T_{f,p}, T_{g,p}] = p^{-1} T_{\frac{1}{\sqrt{-1}}\{f,g\},p} + \mathscr{O}(p^{-2}). \tag{5.6}$$

Remark 13. *For any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ we can consider the linear operator*

$$T_{f,p} : \boldsymbol{L}^2(X, L^p \otimes E) \longrightarrow \boldsymbol{L}^2(X, L^p \otimes E)\,, \quad T_{f,p} = P_p\, f\, P_p\,. \tag{5.7}$$

Then (5.4) *holds for any $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ which are constant outside some compact set. Moreover,* (5.5), (5.6) *still hold for $f, g \in \mathscr{C}^\infty_{const}(X) \subset \mathscr{C}^\infty(X, \mathrm{End}(E))$.*

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Hua operators and Poisson transform for non-tube bounded symmetric domains

Khalid Koufany and Genkai Zhang

1 Introduction

The purpose of the present note is to give an overview of the main results obtained in [9]. Due to the limitation of the space we will be rather brief and descriptive.

Suppose Ω is a bounded symmetric domain in a complex $n-$dimensional space V. Let S be its Shilov boundary and r its rank. We consider the characterization of the image of the Poisson transform $\mathcal{P}_s$ $(s \in \mathbb{C})$ on the Shilov boundary S. For a specific value of s ($s = 1$ in our parameterization) the corresponding Poisson transform $\mathcal{P} := \mathcal{P}_1$ maps hyperfunctions on S to harmonic functions on Ω. When Ω is a tube domain Johnson and Korányi [7] proved that the image of the Poisson transform $\mathcal{P}$ is exactly the set of all Hua-harmonic functions. Lassalle [10] has shown that the Hua system of Johnson and Korányi can be replaced by and equivalent system containing fewer equations. For non-tube domains the characterization of the image of the Poisson transform $\mathcal{P}$ was done by Berline and Vergne [1] where certain third-order differential Hua-operator was introduced to characterize the image. In his paper [15] Shimeno considered the Poisson transform $\mathcal{P}_s$ on tube domains, it is proved that Poisson transform maps hyperfunctions on the Shilov boundary to certain solution space of the Hua operator. For the Shilov boundary of a non-tube domain the problem is still open. We will construct two Hua operators of third order and use them to give a characterization.

2 Notations

Let $\Omega = G/K$ be a bounded symmetric domain of non-tube type in a complex $n-$dimensional vector space V. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra of G. It is known that V carries a unique Jordan triple structure $V \times \bar{V} \times V \to V \ : \ (u, \bar{v}, w) \mapsto \{u\bar{v}w\}$ such that $\mathfrak{p} = \{\xi_v \ : \ v \in V\}$ where $\xi_v(z) = v - Q(z)\bar{v}$ and $Q(z)\bar{v} = \{z\bar{v}z\}$. Let (r, a, b) be the characteristic parameters of the bounded symmetric domain Ω, then $n = rb + r + \frac{r(r-1)}{2}a$. Fix a Jordan frame $\{c_j\}_{j=1}^r$, then $\mathfrak{a} = \oplus \mathbb{R}\xi_{c_j}$ is a maximal Abelian subspace of $\mathfrak{p}$. The restricted roots system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ consists of the roots $\pm\beta_j$ $(1 \le j \le r)$ with multiplicity 1, the roots $\pm\frac{1}{2}\beta_j \pm \frac{1}{2}\beta_k$ $(1 \le j \ne k \le r)$ with multiplicity a, and the roots $\pm\frac{1}{2}\beta_j$ $(1 \le j \le r)$ with multiplicity $2b$. The half sum of the positive roots, is given by $\rho = \sum_{j=1}^r \rho_j\beta_j$, where $\rho_j = \frac{b+1+a(j-1)}{2}$, $j = 1, \ldots, r$. Let

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$\mathfrak{n}^{\pm} = \sum_{\beta \in \Sigma^{\pm}} \mathfrak{g}^{\beta}$ and let $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Let M, A and N be the analytic subgroups of G of Lie algebras $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}^-$ respectively. The subgroup $P = P_{\min} = MAN$ is parabolic subgroup of G and maximal boundary (the Furstenberg boundary) G/P of Ω can be viewed as K/M. Recall that $e = c_1 + c_2 + \ldots + c_r$ is a maximal tripotent of V and the $G-$orbit $S = G \cdot e$ is the minimal boundary (the Shilov boundary) of Ω.

3 The Poisson transform

Let $\mathcal{D}(\Omega)^G$ be the algebra of all invariant differential operators on Ω. Recall the definition of the Harish-Chandra $e_\lambda-$function : e_λ, for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ is the unique $N-$invariant function on Ω such that $e_\lambda(\exp \sum_{j=1}^r t_j \xi_{c_j} \cdot 0) = e^{2\sum_{j=1}^r t_j(\lambda_j + \rho_j)}$. Then e_λ are the eigenfunctions of $T \in \mathcal{D}(\Omega)^G$ and we denote $\chi_\lambda(T)$ the corresponding eigenvalues. Denote further $\mathcal{M}(\Omega, \lambda) = \{f \in C^\infty(\Omega); Tf = \chi_\lambda(T)f, T \in \mathcal{D}(\Omega)^G\}$.

Corresponding to the minimal parabolic subgroup P there is the Poisson transform on the maximal boundary $G/P = K/M$. For $\lambda \in \mathfrak{a}a^*_{\mathbb{C}}$, the *Poisson transform* $\mathcal{P}_{\lambda,K/M}$ is defined by

$$\mathcal{P}_{\lambda,K/M} f(gK) = \int_K e_\lambda(k^{-1}g) f(k) dk$$

on the space $\mathcal{B}(K/M)$ of hyperfunctions on K/M.

It is proved by Kashiwara *et al.* in [8] that for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, if $-2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \{1, 2, 3, \ldots\}$ for all $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$, then the Poisson transform $\mathcal{P}_{\lambda,K/M}$ is a G-isomorphism from $\mathcal{B}(K/M)$ onto $\mathcal{M}(\Omega, \lambda)$.

We now introduce the Poisson transform on the Shilov boundary. Let $h(z)$ be the unique $K-$invariant polynomial on V whose restriction to $\oplus_{j=1}^r \mathbb{R}c_j$ is given by $h(\sum_{j=1}^r t_j c_j) = \prod_{j=1}^r (1 - t_j^2)$. As h is real-valued, we may polarize it to get a polynomial on $V \times V$, denoted by $h(z, w)$, holomorphic in z and anti-holomorphic in w such that $h(z, z) = h(z)$. The *Poisson kernel* $P(z, u)$ on $\Omega \times S$ is $P(z, u) = \Big(h(z, z)/|h(z, u)|^2\Big)^{\frac{n}{r}}$. For a complex number s we define the *Poisson transform* $\mathcal{P}_s$ is defined by

$$(\mathcal{P}_s \varphi)(z) = \int_S P(z, u)^s \varphi(u) d\sigma(u)$$

on the space $\mathcal{B}(S)$ of hyperfunctions on S.

The kernel $P(z, u)^s$, for $u = e$ is a special case of the $e_\lambda-$function. The Poisson transform $\mathcal{P}_s$ on S can be viewed as a restriction of the Poisson transform $\mathcal{P}_{\lambda,K/M}$. For $s \in \mathbb{C}$, we choose a corresponding $\lambda = \lambda_s$ that satisfies the (non-integer) condition of the Kashiwara *et al.* theorem. More precisely we let $\lambda_s = \rho + n(s-1)\xi_c^*$ where $\xi_c^* \in \mathfrak{a}$ such that $\xi_c^*(\xi_c) = 1$ and $\xi_c^*(\xi_c^\perp) = 0$ with $\xi_c = \sum_{j=1}^r \xi_{c_j}$. Then we have an equivalent (non-integer) condition ([9, (1)]) for s. Moreover, $\mathcal{P}_s f(z) = \mathcal{P}_{\lambda_s,K/M} f(z)$ where f on S is viewed as a function on K and thus on K/M. Thus $\mathcal{P}_s\mathcal{B}(S) \subset \mathcal{P}_{\lambda_s,K/M}\mathcal{B}(K/M) \subset \mathcal{M}(\Omega, \lambda_s)$ and when s satisfies ([9, (1)]), $\mathcal{P}_{\lambda_s,K/M}\mathcal{B}(K/M) = \mathcal{M}(\Omega, \lambda_s)$.

4 Hua operators

We define some Hua operators of second and third orders by using the covariant Cauchy-Riemann operator studied in [2], [17] and [19]. Recall briefly that the covariant Cauchy-Riemann operator can be defined on any holomorphic Hermitian vector bundle over a Kähler manifold. Trivializing the sections of a homogeneous vector bundle E on the bounded symmetric space Ω as the space $C^\infty(\Omega, E)$ of E-valued functions on Ω, where E is a holomorphic representation of $K_{\mathbb{C}}$, the *covariant Cauchy-Riemann operator* $\bar{\mathbf{D}}$ is defined by

$$\bar{\mathbf{D}} f = b(z,\bar{z})\bar{\partial} f, \tag{4.1}$$

where $b(z,\bar{z})$ is the inverse of the Bergman metric, also called *Bergman operator* of Ω, given by $b(z,\bar{w}) = 1 - D(z,\bar{w}) + Q(z)Q(\bar{w})$. The operator $\bar{\mathbf{D}}$ maps $C^\infty(\Omega, E)$ to $C^\infty(\Omega, V\otimes E)$, with V viewed as the holomorphic tangent space.

4.1 The second order Hua operator

Consider the space $C^\infty(\Omega)$ of C^∞-functions on Ω as the sections of the trivial line bundle. The operator ∂ is then well-defined on $C^\infty(\Omega)$ and it maps $C^\infty(\Omega)$ to $C^\infty(\Omega, V') = C^\infty(\Omega, \bar{V})$ with the later identified as the space of sections of the holomorphic cotangent bundle. We can then define the differential operator

$$\mathrm{Ad}_{V\otimes\bar{V}}(\bar{\mathbf{D}}\otimes\partial) : C^\infty(\Omega) \to C^\infty(\Omega, \mathfrak{k}_{\mathbb{C}}), \qquad f \mapsto \mathrm{Ad}_{V\otimes\bar{V}}(\bar{\mathbf{D}}\otimes\partial f)$$

with $\mathrm{Ad}_{V\otimes\bar{V}} : V\otimes\bar{V} = \mathfrak{p}^+\otimes\mathfrak{p}^- \to \mathfrak{k}_{\mathbb{C}}$ being the Lie bracket, $u\otimes v \to D(u,v)$. So by the covariant property of ∂ and $\bar{\mathbf{D}}$ (see [19]) we have $\mathrm{Ad}_{V\otimes\bar{V}}(\bar{\mathbf{D}}\otimes\partial)(f(gz)) = dg(z)^{-1}\mathrm{Ad}_{V\otimes\bar{V}}(\bar{\mathbf{D}}\otimes\partial f)(gz)$, where $dg(z) : V = T_z^{(1,0)} \to T_{gz}^{(1,0)}$ is the differential of the mapping g, which further is $dg(z) = \mathrm{Ad}(dg(z))$ the adjoint action of $dg(z)\in K_{\mathbb{C}}$ on $\mathfrak{k}_{\mathbb{C}}$. It follows easily that this operator agree with the Hua operator $\mathcal{H}$ introduced by Johsnon and Korànyi [7],

$$\mathrm{Ad}_{V\otimes\bar{V}}(\bar{\mathbf{D}}\otimes\partial) = \mathcal{H}.$$

Symbolically we may write $\mathcal{H} = D(b(z,\bar{z})\bar{\partial}, \partial)$.

4.2 Third-order Hua operators

Let again E be a homogeneous holomorphic vector bundle on Ω. On E there is a Hermitian structure defined by using the Bergman operator $b(z,\bar{z})$ as an element in $K_{\mathbb{C}}$ and thus there exists a unique Hermitian connection $\nabla : C^\infty(\Omega, E) \to C^\infty(\Omega, T'\otimes E)$, compatible with the complex structure of the cotangent bundle T'. Under the decomposition $T'_z = (T')_z^{(1,0)} + (T')_z^{(0,1)}$ we have $\nabla = \nabla^{(1,0)} + \bar{\partial}$ with

$$\nabla^{(1,0)} : C^\infty(\Omega, E) \to C^\infty(\Omega, (T')^{(1,0)}\otimes E) = C^\infty(\Omega, \mathfrak{p}^-\otimes E),$$

using our identification that $(T')_z^{(1,0)} = \mathfrak{p}^-$. Note that on the space $C^\infty(\Omega)$ of sections of the trivial bundle we have $\nabla^{(1,0)} = \partial$.

We now define two covariant third-order Hua operators $\mathcal{W}$ and $\mathcal{U}$ on $C^\infty(\Omega, E)$ by

$$\mathcal{W} f = \mathrm{Ad}_{\mathfrak{p}^+\otimes\mathfrak{k}_{\mathbb{C}}}\left(\bar{\mathbf{D}}(\mathrm{Ad}_{\mathfrak{p}^+\otimes\mathfrak{p}^-}(\bar{\mathbf{D}}\nabla^{(1,0)} f))\right),$$

$$\mathcal{U} f = \mathrm{Ad}_{\mathfrak{k}_{\mathbb{C}}\otimes\mathfrak{p}^+}\left(\mathrm{Ad}_{\mathfrak{p}^-\otimes\mathfrak{p}^+}(\nabla^{(1,0)}\bar{\mathbf{D}})\bar{\mathbf{D}} f\right).$$

These operators can also be defined by using the enveloping algebra (see [9, Section 7]). For our purpose we consider E to be the trivial representation and the space $C^\infty(\Omega)$ of smooth functions identified as right K-invariant functions on G.

The third-order Hua operator defined by Berline and Vergne in [1] can be viewed, in our context, up to some non-zero constant as

$$\mathcal{V} = \mathrm{Ad}_{\mathfrak{p}^-\otimes\mathfrak{k}_{\mathbb{C}}\to\mathfrak{p}^-}\left(\nabla^{(1,0)}\mathrm{Ad}_{\mathfrak{p}^+\otimes\mathfrak{p}^-\to\mathfrak{k}_{\mathbb{C}}}(\bar{\mathbf{D}}\nabla^{(1,0)})\right).$$

So it is different from our $\mathcal{W}$ and $\mathcal{U}$. For explicit computations the operators $\mathcal{W}$ and $\mathcal{U}$ are somewhat easier to handle as the operator $\bar{\mathbf{D}}$ has a rather explicit formula (4.1) on different holomophic bundles [2], whereas the formula for $\nabla^{(1,0)}$ depends on the metric on the bundles [19]. Note also that the first $\nabla^{(1,0)}$ and the second $\nabla^{(1,0)}$ in $\mathcal{V}$ are different as they are acting on different bundles.

5 Main results

The Hua operator of second-order $\mathcal{H}$ for a general symmetric domain is defined as a $\mathfrak{k}_{\mathbb{C}}$-valued operator. For tube domains it maps the Poisson kernels into the centre of $\mathfrak{k}_{\mathbb{C}}$, namely the Poisson kernels are its eigenfunctions up to an element in the center Z_0, but it is not true for non-tube domains. More precisely we have :

Theorem 5.1 ([9, Theorem 5.3]). *For u fixed in S, the function $z \mapsto P(z,u)^s$ satisfies the following differential equation*

$$\mathcal{H}P(z,u)^s = P(z,u)^s\big[(\frac{n}{r}s)^2 D(b(z,\bar{z})(z^{\bar{z}} - u^{\bar{z}}), \bar{z}^z - \bar{u}^z) - (\frac{n}{r}sp)Z_0\big],$$

where p is the genus of Ω and where x^y denotes the quasi-inverse of x with respect to y.

However for type $\mathbf{I} = \mathbf{I}_{r,r+b}$ domains of non-tube type, there is a variant of the Hua operator, $\mathcal{H}^{(1)}$ see [9, Section 6.], by taking the first component of the operator, since in this case $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}^{(1)} + \mathfrak{k}_{\mathbb{C}}^{(2)}$ is a sum of two irreducible ideals. We prove that the operator $\mathcal{H}^{(1)}$ has the Poisson kernels as its eigenfunctions and we find the eigenvalues. We prove further that the eigenfunctions of the Hua operator $\mathcal{H}^{(1)}$ are also eigenfunctions of invariant differential operators on Ω. For that purpose we compute the radial part of the Hua operator $\mathcal{H}^{(1)}$, see [9, Proposition 6.3]. We give eventually the characterization of the image of the Poisson transform in terms of the Hua operator for type $\mathbf{I}_{r,r+b}$ domains :

Theorem 5.2 ([9, Theorem 6.1]). *Suppose $s \in \mathbb{C}$ satisfies the following condition*

$$-4[b+1+j+(r+b)(s-1)] \notin \{1,2,3,\cdots\}, \;\; \textit{for} \;\; j = 0 \textit{ and } 1.$$

A smooth function f on $\mathbf{I}_{r,r+b}$ is the Poisson transform $\mathcal{P}_s(\varphi)$ of a hyperfunction φ on S if and only if

$$\mathcal{H}^{(1)}f = (r+b)^2 s(s-1) f I_r.$$

Our method of proving the characterization is the same as that in [10] by proving that the boundary value of the Hua eigenfunctions satisfy certain differential equations and are thus defined only on the Shilov boundary, nevertheless it requires several technically

demanding computations.
In [9, Section 7] we study the characterization of range of the Poisson transform for general non-tube domains using the third-order Hua-type operators $\mathcal{U}$ and $\mathcal{W}$:

Theorem 5.3 ([9, Theorem 7.2]). *Let Ω be a bounded symmetric non-tube domain of rank r in $\mathbb{C}^n$. Let $s \in \mathbb{C}$ and put $\sigma = \frac{n}{r}s$. If a smooth function f on Ω is the Poisson transform $\mathcal{P}_s$ of a hyperfunction in $\mathcal{B}(S)$, then*

$$\left(\mathcal{U} - \frac{-2\sigma^2 + 2p\sigma + c}{\sigma(2\sigma - p - b)}\mathcal{W}\right) f = 0. \tag{5.1}$$

Conversely, suppose s satisfies the condition

$$-4[b + 1 + j\frac{a}{2} + \frac{n}{r}(s-1)] \notin \{1, 2, 3, \cdots\}, \quad for \;\; j = 0 \; and \; 1.$$

Let f be an eigenfunction $f \in \mathcal{M}(\Omega, \lambda_s)$. If f satisfies (5.1) then it is the Poisson transform $\mathcal{P}_s(\varphi)$ of a hyperfunction φ on S.

After this paper was finished we were informed by Professor T. Oshima that he and N. Shimeno have obtained some similar results about Poisson transforms and Hua operators.

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Unitarizability of holomorphically induced representations of a split solvable Lie group

Hideyuki Ishi

1 Introduction

Let G be a connected and simply connected split solvable Lie group with the Lie algebra $\mathfrak{g}$, and $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ a totally complex positive polarization at a linear form λ on $\mathfrak{g}$. In this article, we consider the unitarizability of a representation of G holomorphically induced from $\mathfrak{h}$. The subject is an analogue of the unitarizability of the highest weight representations of a Hermitian Lie group ([4], [11], [16], [17]).

We state the settings more precisely. The stabilizer

$$G_\lambda := \{\, g \in G\,;\, \mathrm{Ad}^*(g)\lambda = \lambda \,\}$$

at λ equals the subgroup $\exp(\mathfrak{h} \cap \mathfrak{g})$ of G. Let ν_λ be a unitary character of G_λ given by $\nu_\lambda(\exp X) := e^{i\langle X, \lambda\rangle}$ $(X \in \mathfrak{h} \cap \mathfrak{g})$. For $X \in \mathfrak{g}$ and $\phi \in C^\infty(G)$, we define $R(X)\phi \in C^\infty(G)$ by $R(X)\phi(a) := \left(\frac{d}{dt}\right)_{t=0} \phi(a \exp tX)$ $(a \in G)$, and for $X_1, X_2 \in \mathfrak{g}$, define $R(X_1 + iX_2)\phi := R(X_1)\phi + iR(X_2)\phi$. We denote by $\mathcal{C}(G, \mathfrak{h}, \lambda)$ the space of smooth functions ϕ on G satisfying the conditions (C1) $\phi(xh) = \nu_\lambda(h)^{-1}\phi(x)$ $(x \in G,\, h \in G_\lambda)$, and (C2) $R(Z)\phi = -i\langle Z, \lambda\rangle\phi$ $(Z \in \mathfrak{h})$, where λ is extended complex linearly to $\mathfrak{g}_{\mathbb{C}}$. Our holomorphically induced representation τ_λ is defined on $\mathcal{C}(G, \mathfrak{h}, \lambda)$ by the left translation: $\tau_\lambda(g)\phi(x) := \phi(g^{-1}x)$ $(g, x \in G)$. Put

$$\mathcal{H}^2(G, \mathfrak{h}, \lambda) := \left\{ \phi \in \mathcal{C}(G, \mathfrak{h}, \lambda)\,;\, \|\phi\|^2 := \int_{G/G_\lambda} |\phi(g)|^2 \, d\dot{g} < +\infty \right\},$$

where $d\dot{g}$ denotes an invariant measure on the coset space G/G_λ. If $\mathcal{H}^2(G, \mathfrak{h}, \lambda)$ is non-trivial, $\mathcal{H}^2(G, \mathfrak{h}, \lambda)$ is a G-invariant Hilbert space and the representation $(\tau_\lambda, \mathcal{H}^2(G, \mathfrak{h}, \lambda))$ is unitary. Moreover, $\mathcal{H}^2(G, \mathfrak{h}, \lambda)$ has a reproducing kernel and $(\tau_\lambda, \mathcal{H}^2(G, \mathfrak{h}, \lambda))$ is irreducible. The non-vanishing condition of $\mathcal{H}^2(G, \mathfrak{h}, \lambda)$ is given by Fujiwara ([5], [6]). Now we note that, even if $\mathcal{H}^2(G, \mathfrak{h}, \lambda) = \{0\}$, there may still exist a non-zero G-invariant subspace $\mathcal{H}(G, \mathfrak{h}, \lambda)$ of $\mathcal{C}(G, \mathfrak{h}, \lambda)$ with a reproducing kernel Hilbert space structure such that $(\tau_\lambda, \mathcal{H}(G, \mathfrak{h}, \lambda))$ is a unitary representation of G. The representation $(\tau_\lambda, \mathcal{H}(G, \mathfrak{h}, \lambda))$ is necessarily irreducible by [12]. We shall describe the condition for the existence of such $\mathcal{H}(G, \mathfrak{h}, \lambda)$ (Theorem 15), and determine the coadjoint orbit in $\mathfrak{b}^*$ corresponding to the representation $(\tau_\lambda, \mathcal{H}(G, \mathfrak{h}, \lambda))$ by the Kirillov-Bernat correspondence [2]. In a certain case, we realize the Hilbert space as a space of holomorphic functions on a Siegel domain (section 4).

2 Normal j-algebras and Siegel domains

Since our study is based on the theory of normal j-algebras established by Piatetskii-Shapiro, we first review his results [14, Chapter 2, Sections 3 and 5] briefly. Let $\mathfrak{b}$ be a split solvable Lie algebra, $j : \mathfrak{b} \to \mathfrak{b}$ a linear map such that $j^2 = -\mathrm{id}_{\mathfrak{b}}$, and ω a linear form on $\mathfrak{b}$. The triple $(\mathfrak{b}, j, \omega)$ is called a *normal j-algebra* if the following are satisfied: (NJA1) $[Y_1, Y_2] + j[jY_1, Y_2] + j[Y_1, jY_2] - [jY_1, jY_2] = 0$ for all $Y_1, Y_2 \in \mathfrak{b}$, and (NJA2) the bilinear form $(Y_1|Y_2)_\omega := \langle [Y_1, jY_2], \omega \rangle$ $(Y_1, Y_2 \in \mathfrak{b})$ gives a j-invariant inner product on $\mathfrak{b}$. Let $\mathfrak{a}$ be the orthogonal complement of the subspace $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}$ with respect to $(\cdot|\cdot)_\omega$. Then $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{b}$. Put $r := \dim \mathfrak{a}$, and for a linear form $\alpha \in \mathfrak{a}^*$, set $\mathfrak{b}_\alpha := \{\, Y \in \mathfrak{b}; [C, Y] = \langle C, \alpha \rangle Y \ (C \in \mathfrak{a}) \}$.

Proposition 2.1 (Piatetskii-Shapiro). (i) *There is a linear basis $\{A_1, \dots, A_r\}$ of $\mathfrak{a}$ such that if one puts $E_l := -jA_l$, then $[A_k, E_l] = \delta_{kl} E_l$ $(k, l = 1, \dots, r)$.*
(ii) *Let $\alpha_1, \dots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $A_1, \dots, A_r$. Then one has a decomposition $\mathfrak{b} = \mathfrak{b}(1) \oplus \mathfrak{b}(1/2) \oplus \mathfrak{b}(0)$ with*

$$\mathfrak{b}(1) := \sum_{k=1}^{r}{}^{\oplus} \mathbb{R}E_k \oplus \sum_{1 \le k < m \le r}^{\oplus} \mathfrak{b}_{(\alpha_m + \alpha_k)/2}, \quad \mathfrak{b}(1/2) := \sum_{k=1}^{r}{}^{\oplus} \mathfrak{b}_{\alpha_k/2},$$
$$\mathfrak{b}(0) := \mathfrak{a} \oplus \sum_{1 \le k < m \le r}^{\oplus} \mathfrak{b}_{(\alpha_m - \alpha_k)/2}.$$

(iii) *One has $[\mathfrak{b}(p), \mathfrak{b}(q)] \subset \mathfrak{b}(p+q)$ $\quad (p, q = 0, 1/2, 1)$, where $\mathfrak{b}(p) := \{0\}$ if $p > 1$.* (iv) *One has $j\mathfrak{b}_{(\alpha_m - \alpha_k)/2} = \mathfrak{b}_{(\alpha_m + \alpha_k)/2}$ $(1 \le k < m \le r)$ and $j\mathfrak{b}_{\alpha_k/2} = \mathfrak{b}_{\alpha_k/2}$ $(k = 1, \dots, r)$.*

Let B be the connected and simply connected Lie group corresponding to $\mathfrak{b}$. The group B is realized as an affine transformation group acting on a Siegel domain D constructed as follows. The subgroup $B(0) := \exp \mathfrak{b}(0)$ of B acts on $\mathfrak{b}(1)$ by the adjoint action because of Proposition 2.1 (iii). Let Ω be the $B(0)$-orbit through $E := E_1 + \cdots + E_r$ in $\mathfrak{b}(1)$. Then Ω is a regular open convex cone on which $B(0)$ acts simply transitively. By Proposition 2.1 (iv), the operator j defines a complex structure on $\mathfrak{b}(1/2)$. We define a $\mathfrak{b}(1)_{\mathbb{C}}$-valued Hermitian map Q on $(\mathfrak{b}(1/2), j)$ by $Q(u, u') := ([ju, u'] + i[u, u'])/4$ $(u, u' \in \mathfrak{b}(1/2))$. Our Siegel domain D is a complex domain in $\mathfrak{b}(1)_{\mathbb{C}} \times (\mathfrak{b}(1/2), j)$ defined by $D := \{\, (z, u)\, ;\, \Im z - Q(u, u) \in \Omega \,\}$. The group B acts on D simply transitively by

$$\begin{aligned}
&\exp(x_0 + u_0) h_0 \cdot (z, u) \\
&\quad := (\mathrm{Ad}(h_0) z + x_0 + 2iQ(\mathrm{Ad}(h_0) u, u_0) + iQ(u_0, u_0), \mathrm{Ad}(h_0) u + u_0) \\
&\qquad\qquad (x_0 \in \mathfrak{b}(1),\ u_0 \in \mathfrak{b}(1/2),\ h_0 \in B(0),\ (z, u) \in D).
\end{aligned}$$

Let $\mathfrak{b}_-$ be the subspace $\{\, Y + ijY\, ;\, Y \in \mathfrak{b} \,\}$ of $\mathfrak{b}_{\mathbb{C}}$. Then $\mathfrak{b}_-$ is a totally complex positive polarization at $-\omega \in \mathfrak{b}^*$ ([15]). Let $\mathfrak{b}'$ be the orthogonal complement of the one dimensional subspace $\mathbb{R}A_r$ in $\mathfrak{b}$, and ω' the restriction of ω to $\mathfrak{b}'$. Put $\mathfrak{b}'_- := \mathbb{C}E_r \oplus (\mathfrak{b}'_{\mathbb{C}} \cap \mathfrak{b}_-) \subset \mathfrak{b}'_{\mathbb{C}}$. Then $\mathfrak{b}'_-$ is a totally complex positive polarization at $-\omega'$.

3 Non-vanishing condition of $\mathcal{H}(G, \mathfrak{h}, \lambda)$

Now recall the polarization $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ at $\lambda \in \mathfrak{g}^*$ in section 1.

Theorem 14. *For the triple* $(\mathfrak{g}, \mathfrak{h}, \lambda)$*, there exists a normal* j*-algebra* $(\mathfrak{b}, j, \omega)$ *and a Lie algebra homomorphism* $\varpi : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{b}_{\mathbb{C}}$ *satisfying either*

$$\varpi(\mathfrak{g}) = \mathfrak{b}, \quad \varpi(\mathfrak{h}) = \mathfrak{b}_-, \quad \lambda = -\omega \circ \varpi, \tag{3.1}$$

or

$$\varpi(\mathfrak{g}) = \mathfrak{b}', \quad \varpi(\mathfrak{h}) = \mathfrak{b}'_-, \quad \lambda = -\omega' \circ \varpi. \tag{3.2}$$

This $(\mathfrak{b}, j, \omega)$ *is unique up to isomorphisms.*

Let $\tilde{\varpi} : G \to B$ the Lie group homomorphism given by $\tilde{\varpi}(\exp X) := \exp \varpi(X)$ $(X \in \mathfrak{g})$. The pull back $\tilde{\varpi}^*$ induces an isomorphism from either $\mathcal{C}(B, \mathfrak{b}_-, -\omega)$ or $\mathcal{C}(B', \mathfrak{b}'_-, -\omega')$ onto $\mathcal{C}(G, \mathfrak{h}, \lambda)$, where B' is the subgroup $\exp \mathfrak{b}'$ of B. Thus the existence of a non-zero $\mathcal{H}(G, \mathfrak{h}, \lambda)$ is equivalent of the one of $\mathcal{H}(B, \mathfrak{b}_-, -\omega)$ or $\mathcal{H}(B', \mathfrak{b}'_-, -\omega')$, so that we can describe the non-vanishing condition of $\mathcal{H}(G, \mathfrak{h}, \lambda)$ in terms of the structure of the normal j-algebra $(\mathfrak{b}, j, \omega)$. For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r) \in \{0, 1\}^r$ with $\varepsilon_r = 1$, we set $q_k(\varepsilon) := \sum_{m>k} \varepsilon_m \dim \mathfrak{b}_{(\alpha_m - \alpha_k)/2}$ $(k = 1, \dots, r)$ and

$$\mathcal{X}(\varepsilon) := \{\, s \in \mathbb{R}^r \,;\, s_k > q_k(\varepsilon)/4 \text{ (if } \varepsilon_k = 1),\ s_k = q_k(\varepsilon)/4 \text{ (if } \varepsilon_k = 0) \,\}.$$

Let $\mathcal{X}$ be the disjoint union $\bigsqcup_{\varepsilon} \mathcal{X}(\varepsilon)$.

Theorem 15. *Put* $\gamma_k := \langle \omega, E_k \rangle$ $(k = 1, \dots, r)$*. Then a non-zero* $\mathcal{H}(G, \mathfrak{h}, \lambda)$ *exists if and only if* $\gamma := (\gamma_1, \dots, \gamma_r)$ *belongs to* $\mathcal{X}$*.*

Thanks to Proposition 2.1, any element Y of $\mathfrak{b}$ is expressed as $Y = \sum_{k=1}^r (c_k A_k + x_{kk} E_k) + Y_0$ with c_k, $x_{kk} \in \mathbb{R}$ and $Y_0 \in (\mathfrak{a} \oplus j\mathfrak{a})^\perp$. Then we have $\langle Y, \omega \rangle = \sum_{k=1}^r (\beta_k c_k + \gamma_k x_{kk})$, where $\beta_k := \langle A_k, \omega \rangle$ $(k = 1, \dots, r)$.

Theorem 16. *If* γ *belongs to* $\mathcal{X}(\varepsilon)$*, define* $\omega_\varepsilon \in \mathfrak{b}^*$ *by* $\langle Y, \omega_\varepsilon \rangle := \sum_{k=1}^r (\beta_k c_k + \varepsilon_k \gamma_k x_{kk})$*. Then the irreducible unitary representation* $(\tau_\lambda, \mathcal{H}(G, \mathfrak{h}, \lambda))$ *corresponds to the coadjoint orbit through* $\lambda_\varepsilon := -\omega_\varepsilon \circ \varpi \in \mathfrak{g}^*$ *by the Kirillov-Bernat correspondence.*

We remark that λ_ε belongs to the boundary of the coadjoint orbit $\mathrm{Ad}(G)^*\lambda \subset \mathfrak{g}^*$ unless $\varepsilon = (1, \dots, 1)$.

4 Function spaces on the Siegel domain D

Rossi and Vergne [15] observed that the spaces $\mathcal{C}(B, \mathfrak{b}_-, -\omega)$ and $\mathcal{H}^2(B, \mathfrak{b}_-, -\omega)$ are realized as spaces of holomorphic functions on the Siegel domain D (see also [6, Section 5B]). Thanks to this fact, Theorems 15 and 16 are deduced from results about analysis on Siegel domains [10] in the case (3.1) in Theorem 14. In this section, we give a similar realization of $\mathcal{C}(B', \mathfrak{b}'_-, -\omega)$ and $\mathcal{H}(B', \mathfrak{b}'_-, -\omega)$ as function spaces on D. The results play substantial roles in the study of the case (3.2).

We define a one-dimensional representation $\chi_\omega : B \to \mathbb{C}$ of B in such a way that $\chi_\omega(\exp C) = e^{i\langle C + ijC, \omega \rangle}$ $(C \in \mathfrak{a})$. Set $\kappa := \langle E_r, \omega \rangle$. We denote by $\mathcal{O}(D; \kappa)$ the space of holomorphic functions F on D with the property $F(z + cE_r, u) = e^{i\kappa c} F(z, u)$ $((z, u) \in D,\ c \in \mathbb{R})$. For $F \in \mathcal{O}(D; \kappa)$, define a function $\Psi_\omega F$ on the group B' by $\Psi_\omega F(b) := \chi_\omega(b) F(b \cdot p_0)$ $(b \in B')$, where $p_0 := (iE, 0) \in D$.

Lemma 4.1. *The map* Ψ_ω *gives an isomorphism from* $\mathcal{O}(D; \kappa)$ *onto* $\mathcal{C}(B', \mathfrak{b}'_-, -\omega')$*.*

Recalling Theorem 15, we assume that $\gamma \in \mathcal{X}(\varepsilon)$. Let $\mathcal{H}_\omega(D)$ be a subspace of $\mathcal{O}(D;\kappa)$ with a Hilbert space structure such that Ψ_ω gives a unitary isomorphism from $\mathcal{H}_\omega(D)$ onto $\mathcal{H}(B', \mathfrak{b}'_-, -\omega')$. Put $B'(0) := B(0) \cap B'$. Then each $x \in \Omega$ is uniquely expressed as $x = \mathrm{Ad}(h)E + cE_r$ with $h \in B'(0)$ and $c > -1$. Let Υ_ω be a function on Ω defined by $\Upsilon_\omega(\mathrm{Ad}(h)E + cE_r) := e^{-\kappa c}|\chi_\omega(h)|^{-2}$. We denote by $\xi_\varepsilon \in \mathfrak{b}(1)^*$ the restriction of ω_ε to $\mathfrak{b}(1)$ (see Theorem 16), and by $\mathcal{O}^*$ the $B'(0)$-orbit $\mathrm{Ad}^*(B'(0))\xi_\varepsilon \subset \mathfrak{b}(1)^*$.

Proposition 4.2. *There exists a $B'(0)$-relatively invariant measure $d\mu_\omega$ on $\mathcal{O}^*$ such that*

$$\Upsilon_\omega(x) = \int_{\mathcal{O}^*} e^{-\langle x,\xi\rangle}\, d\mu_\omega(\xi) \quad (x \in \Omega).$$

The function Υ_ω is analytically continued to a holomorphic function on the domain $\Omega + i\mathfrak{b}(1) \subset \mathfrak{b}(1)_{\mathbb{C}}$ by $\Upsilon_\omega(z) = \int_{\mathcal{O}^*} e^{-\langle z,\xi\rangle}\, d\mu_\omega(\xi) \quad (z \in \Omega + i\mathfrak{b}(1))$.

Proposition 4.3. *The reproducing kernel K^ω of $\mathcal{H}_\omega(D)$ is given by*

$$K^\omega((z_1,u_1),(z_2,u_2)) = A\Upsilon_\omega((z_1 - \bar{z}_2)/i - 2Q(u_1,u_2)) \quad ((z_1,u_1),(z_2,u_2) \in D),$$

where A is a constant.

We normalize the inner product of $\mathcal{H}_\omega(D)$ so that $A = 1$. Now we give a Paley-Wiener type description of the Hilbert space $\mathcal{H}_\omega(D)$. For $\xi \in \mathcal{O}^*$, let $\mathcal{F}_\xi$ be the Fock-Bargmann space on $(\mathfrak{b}(1/2), j)$ whose reproducing kernel is $e^{2\xi\circ Q}$.

Theorem 17. *One has a unitary isomorphism $\Phi_\omega : \int_{\mathcal{O}^*}^{\oplus} \mathcal{F}_\xi d\mu_\omega(\xi) \ni f \mapsto F \in \mathcal{H}_\omega(D)$, where*

$$F(z,u) := \int_{\mathcal{O}^*} e^{i\langle z,\xi\rangle} f(\xi)(u)\, d\mu_\omega(\xi) \quad ((z,u) \in D).$$

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