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Berezin Transform for Solvable Groups ⋆

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Abstract.

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0. Introduction

The Berezin transform for complex multivariable domains $D \subset \mathbb{C}^n$ is important to harmonic analysis because of its covariance with respect to holomorphic transformations. It can be regarded as an analogue of the Poisson transform, replacing the boundary integration by integrating over the domain itself. This applies in particular to *homogeneous* domains where a Lie group G of holomorphic transformations acts transitively on D. The best known class are the *symmetric* domains where G can be chosen as a *semi-simple* Lie group and the quantization Hilbert spaces are the weighted Bergman spaces $H^2_\nu(D)$ of holomorphic functions on D, where ν is a scalar parameter. For more general homogeneous domains one chooses instead a *solvable* Lie group G, with corresponding 'multi-weighted' Bergman space $H^{\circ 2}(D)$ depending on a vector parameter $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_r)$. This case has been studied in detail by Gindikin [4]. In this paper we study the Berezin transform for the multi-weighted Bergman spaces of Gindikin type. In the semi-simple case of symmetric domains $D = G/K$, the Plancherel decomposition of $L^2(D)$ expresses the Berezin transform in terms of eigenvalues which are known explicitly [6], with precise information on the asymptotic behaviour. For solvable groups $G = \mathrm{AN}$ the Plancherel decomposition is quite different, it involves the discrete series of AN and, accordingly, the 'spectral components' of the Berezin transform $\mathcal{B}$ are not eigenvalues but operators on suitable representation Hilbert spaces. In addition, the nonunimodularity of G results in a modification of the Plancherel decomposition which has to be addressed in the spectral analysis of $\mathcal{B}$. We obtain an explicit realization of the spectral component of $\mathcal{B}$, expressing its integral kernel in closed

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form as a multi-variable hypergeometric function, and prove the correspondence principle for the Berezin transform in a general setting of summability theory.

1. Solvable Groups Associated with Siegel Domains

Let X be a real vector space of finite dimension d_1 and let $\Omega \subset X$ be a sharp open convex cone. Then

$$\mathrm{GL}(\Omega) := \{g \in \mathrm{GL}(X) : g\Omega = \Omega\}$$

is a closed subgroup of $\mathrm{GL}(X)$ and therefore a real Lie group. The cone Ω is called *homogeneous* if $\mathrm{GL}(\Omega)$ acts transitively on Ω. A homogeneous open convex cone Ω is called *symmetric* if there exists a scalar product $(x|y)$ on X such that

$$\Omega = \{x \in X : (x|y) > 0 \ \forall y \in \overline{\Omega} \setminus \{0\}\}. \tag{1.1}$$

In this case X becomes a *Euclidean Jordan algebra* [3], with product $x \circ y$ and unit element e, such that

$$\Omega = \{y^2 : y \in X \text{ invertible}\}. \tag{1.2}$$

Let $U := X^{\mathbb{C}}$ be the complexification of X, endowed with the canonical involution

$$(x + iy)^* := x - iy \quad (x, y \in X).$$

Let V be a complex vector space of finite dimension d_2, and let $\Phi\colon V \times V \to U$ be a sesqui-linear map (conjugate-linear in the second variable) such that $\Phi(v, v) \in \Omega$ for all $v \in V \setminus \{0\}$. Then

$$\Phi(v_1, v_2)^* = \Phi(v_2, v_1)$$

for all $v_1, v_2 \in V$. We put $Z := U \times V = X^{\mathbb{C}} \times V$ and define $\tau\colon Z \times Z \to U$ by

$$\tau((u_1, v_1), (u_2, v_2)) := u_1 + u_2^* - \Phi(v_1, v_2)$$

for all $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Then $\tau(z_1, z_2)^* = \tau(z_2, z_1)$ for all $z_1, z_2 \in Z$. The open convex domain

$$\begin{aligned} D = T(\Omega, \Phi) &= \{(u, v) \in U \times V : u + u^* - \Phi(v, v) \in \Omega\} \\ &= \{z \in Z : \tau(z, z) \in \Omega\} \end{aligned}$$

is called the *Siegel domain* associated with Ω and Φ. In the special case $V = \{0\}$, we obtain the *tube domain*

$$\begin{aligned} D = T(\Omega) &= \{u \in U : u + u^* \in \Omega\} \\ &= \{x + iy : x \in \Omega,\ y \in X\} = \Omega + iX \end{aligned}$$

associated with Ω. A tube domain $T(\Omega)$ is invariant under 'imaginary' translations

$$t_a u := u + a \quad (u \in X^{\mathbb{C}},\ a \in iX).$$

In the nontube case, the ‘quasi-translations’

$$t_{a,b}(u,v) := \left(u + a + \Phi(v,b) + \frac{\Phi(b,b)}{2},\ v + b\right),$$

for $a \in iX$, $b \in V$, $u \in X^{\mathbb{C}}$, $v \in V$ satisfy

$$\tau(t_{a,b}z_1, t_{a,b}z_2) = \tau(z_1, z_2)$$

for all $z_1, z_2 \in Z$, and hence leave $D = T(\Omega, \Phi)$ invariant. Since $t_{a,b}^{-1} = t_{-a,-b}$ and

$$t_{a_1,b_1}t_{a_2,b_2} = t_{a_1+a_2+(\Phi(b_2,b_1)-\Phi(b_1,b_2))/2,\ b_1+b_2} \tag{1.3}$$

it follows that

$$\Sigma := \{t_{a,b} : a \in iX,\ b \in V\}$$

is a group of affine transformations of D which is nilpotent of step 2 (generalized Heisenberg group) and is usually identified with the orbit

$$\begin{aligned} \Sigma(0) &= \{t(0) : t \in \Sigma\} \\ &= \{z \in Z : \tau(z,z) = 0\} \\ &= \{(a + \Phi(b,b)/2,\ b) : a \in iX, b \in V\} \end{aligned}$$

which is the *Shilov boundary* of D.

Now suppose Ω is a symmetric cone, realized as the positive cone (1.2) of a Euclidean Jordan algebra X. Then $\mathrm{GL}(\Omega) \subset \mathrm{GL}(X)$ acts on $U = X^{\mathbb{C}}$ by complexification. Moreover, there is a representation $(g, v) \mapsto gv$ of $\mathrm{GL}(\Omega)$ on V satisfying

$$\Phi(gv_1, gv_2) = g\Phi(v_1, v_2) \tag{1.4}$$

for all $v_1, v_2 \in V$. Defining

$$g(u,v) := (gu, gv) \tag{1.5}$$

for $u \in U$, $v \in V$, we have

$$\tau(gz_1, gz_2) = g\tau(z_1, z_2)$$

and therefore the linear transformations (1.5) leave D invariant. Since

$$gt_{a,b}g^{-1} = t_{ga,gb} \tag{1.6}$$

for all $g \in \mathrm{GL}(\Omega)$, $a \in iX$, $b \in V$ it follows that $\mathrm{GL}(\Omega)$ acts on the group Σ, and we may form the *semi-direct product*

$$\mathrm{Aff}(D) = \mathrm{GL}(\Omega) \triangleleft \Sigma = \{(g,t) : g \in \mathrm{GL}(\Omega),\ t \in \Sigma\},$$

consisting of all affine transformations of D, endowed with the product

$$(g_1, t_{a_1,b_1})(g_2, t_{a_2,b_2}) = (g_1 g_2,\ t_{g_1 a_1, g_1 b_1} t_{a_2,b_2})$$

for all $g_1, g_2 \in \mathrm{GL}(\Omega)$, $a_1, a_2 \in iX$ and $b_1, b_2 \in V$.

Now fix a frame $e_1, \ldots, e_r$ of minimal orthogonal projections in X, with $e := e_1 + \cdots + e_r$ the unit element. The associated *Peirce decomposition*

$$X = \sum_{1 \leqslant i \leqslant j \leqslant r} X_{ij}$$

gives rise to a solvable subgroup $\mathrm{AN}_\Omega \subset \mathrm{GL}(\Omega)$ which acts simply transitive on Ω [1]. Similarly, the semi-direct product

$$\mathrm{AN} := \mathrm{AN}_\Omega \lhd \Sigma$$

is a simply transitive solvable subgroup of the biholomorphic automorphism group of D.

For an irreducible Euclidean Jordan algebra X of rank r, the 'Jordan determinant' $\Delta\colon X \to \mathbb{R}$ is a r-homogeneous irreducible polynomial satisfying $\Delta(e) = 1$ and

$$\Delta(gx) = \Delta(ge)\Delta(x)$$

for all $g \in \mathrm{GL}(\Omega)$ and $x \in X$. For any $1 \leqslant k \leqslant r$,

$$X_k := \sum_{1 \leqslant i \leqslant j \leqslant k} X_{ij}$$

is a subalgebra of X of rank k, with unit element $e_1 + \cdots + e_k$. It has a Jordan determinant function Δ_{X_k}, and we define $\Delta_k\colon X \to \mathbb{R}$ by

$$\Delta_k(x) := \Delta_{X_k}(P_k x), \tag{1.7}$$

where $P_k\colon X \to X_k$ is the Peirce 1-projection of $e_1 + \cdots + e_k$. Then $\Delta_r = \Delta$. Using the 'minors' (1.7), we define *conical functions*

$$\Delta_{\mathbf{s}}(x) := \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \cdots \Delta_r(x)^{s_r} \tag{1.8}$$

for all $x \in \Omega$, where $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ is a multi-parameter.

2. Spectral Components of Invariant Operators

For nonunimodular groups G such as AN_Ω and AN (cf. Section 1), the Plancherel formalism has to be slightly modified: Let $\langle G \rangle_\pi$ be an irreducible representation space of a nonunimodular group G, with inner product $(\phi_1|\phi_2)$ conjugate-linear in the first variable, and consider the rank 1 operator

$$(\phi_1 \phi_2^*)\phi := \phi_1(\phi_2|\phi), \quad \phi \in \langle G \rangle_\pi$$

on $\langle G\rangle_\pi$, induced by $\phi_1, \phi_2 \in \langle G\rangle_\pi$. For a left Haar measure $\mathrm{d}\lambda$, the operator

$$T_{\phi_2,\phi_1} := \int_G \mathrm{d}\lambda(g)(\pi(g)\phi_1)(\pi(g)\phi_2)^*$$

on $\langle G\rangle_\pi$ commutes with the action of G, since for all $\gamma \in G$

$$\begin{aligned}\pi(\gamma)T_{\phi_2,\phi_1} &= \int_G \mathrm{d}\lambda(g)(\pi(\gamma g)\phi_1)(\pi(g)\phi_2)^* \\ &= \int_G \mathrm{d}\lambda(g)(\pi(\gamma g)\phi_1)(\pi(\gamma g)\phi_2)^*\pi(\gamma) = T_{\phi_2,\phi_1}\pi(\gamma).\end{aligned}$$

By Schur's lemma, T_{ϕ_2,ϕ_1} is a scalar operator, more precisely

$$S_{\phi_2,\phi_1} = (\phi_2|S\phi_1)\cdot I,$$

where $S = S_\pi$ is a positive (possibly unbounded) operator in $\langle G\rangle_\pi$. Thus

$$(\phi_2|S\phi_1)\psi = \int_G \mathrm{d}\lambda(g)(\pi(g)\phi_2|\psi)\pi(g)\phi_1 \tag{2.1}$$

for all $\phi_1, \phi_2 \in \mathrm{Dom}(S)$ and $\psi \in \langle G\rangle_\pi$. Equivalently

$$(\phi_2|S\phi_1)(\psi_1|\psi_2) = \int_G \mathrm{d}\lambda(g)(\pi(g)\phi_2|\psi_2)(\psi_1|\pi(g)\phi_1). \tag{2.2}$$

Note that the operator S depends on the normalization of λ. For $\phi, \psi \in \mathrm{Dom}\, S^{1/2}$, we define the *modified coefficient function*

$$(\phi \boxtimes \psi)(g) = (\phi|\pi(g)S^{-1/2}\psi), \quad g \in G.$$

Then clearly

$$\ell_g(\phi \boxtimes \psi) = (\pi(g)\phi) \boxtimes \psi,$$

where $(\ell_g f)(\gamma) := f(g^{-1}\gamma)$ denotes left translation. By (2.2), we have

$$(\phi_1 \boxtimes \psi_1|\phi_2 \boxtimes \psi_2)_{L^2(G)} = (\psi_1|\psi_2)(\phi_2|\phi_1) \tag{2.3}$$

for all $\phi_1, \phi_2 \in \langle G\rangle_\pi$, $\psi_1 \in \mathrm{Dom}\, S^{-1/2}$, $\psi_2 \in \mathrm{Dom}\, S^{1/2}$.

Now let $\mathcal{B}\colon L^2(G) \to L^2(G)$ be a densely defined integral operator

$$(\mathcal{B}f)(g) = \int_G \mathrm{d}\lambda(h)\mathcal{B}(g,h)f(h), \quad f \in L^2(G),$$

with left-invariant integral kernel:

$$\mathcal{B}(\gamma g, \gamma h) = \mathcal{B}(g,h)$$

for all $g, \gamma, h \in G$. Then $\mathcal{B}$ commutes with left translations:

$$\begin{aligned}\mathcal{B}(\ell_\gamma f)(g) &= \int_G d\lambda(h)\mathcal{B}(g,h)(\ell_\gamma f)(h) = \int_G d\lambda(h)\mathcal{B}(g,h)f(\gamma^{-1}h)\\ &= \int_G d\lambda(h)\mathcal{B}(\gamma^{-1}g,\gamma^{-1}h)f(\gamma^{-1}h) = (\mathcal{B}f)(\gamma^{-1}g)\\ &= \ell_\gamma(\mathcal{B}f)(g).\end{aligned}$$

For any irreducible representation π of G, define the *spectral component* $\mathcal{B}_\pi$ of $\mathcal{B}$ as the operator

$$\mathcal{B}_\pi := \int_G d\lambda(g)\mathcal{B}(1,g)S^{1/2}\pi(g)S^{-1/2} \tag{2.4}$$

on $\langle G\rangle_\pi$. Here $1 \in G$ is the identity, and S is defined via (2.1).

PROPOSITION 2.1. *For $\phi, \psi \in \langle G\rangle_\pi$ we have*

$$\mathcal{B}(\phi \boxtimes \psi) = \phi \boxtimes (\mathcal{B}_\pi \psi). \tag{2.5}$$

Proof. Putting $\gamma := g^{-1}h$ we have

$$\begin{aligned}\mathcal{B}(\phi\boxtimes\psi)(g) &= \int_G d\lambda(h)\mathcal{B}(g,h)(\phi\boxtimes\psi)(h)\\ &= \int_G d\lambda(h)\mathcal{B}(1,g^{-1}h)(\phi|\pi(h)S^{-1/2}\psi)\\ &= \int_G d\lambda(\gamma)\mathcal{B}(1,\gamma)(\phi|\pi(g)\pi(\gamma)S^{-1/2}\psi)\\ &= (\phi|\pi(g)\int_G d\lambda(\gamma)\mathcal{B}(1,\gamma)\pi(\gamma)S^{-1/2}\psi)\\ &= (\phi|\pi(g)S^{-1/2}\mathcal{B}_\pi\psi) = (\phi\boxtimes(\mathcal{B}_\pi\psi))(g).\end{aligned}$$ □

In view of (2.3), we obtain

COROLLARY 2.2. *For $\phi_1, \phi_2, \psi_1, \psi_2$ we have*

$$(\phi_1 \boxtimes \psi_1|\mathcal{B}(\phi_2 \boxtimes \psi_2))_{L^2(G)} = (\phi_2|\phi_1)(\psi_1|\mathcal{B}_\pi\psi_2). \tag{2.6}$$

As formal consequences of (2.5) and (2.6) we obtain

COROLLARY 2.3. *For left-invariant operators $\mathcal{B}$, $\mathcal{A}$ on $L^2(G)$, we have*

(i) $(\mathcal{B}\mathcal{A})_\pi = \mathcal{B}_\pi\mathcal{A}_\pi$.
(ii) $(\mathcal{B}^*)_\pi = (\mathcal{B}_\pi)^*$.
(iii) *If $\mathcal{B}$ is normal, unitary, self-adjoint, positive, resp., then $\mathcal{B}_\pi$ is also normal, unitary, self-adjoint, positive, resp.*

(iv) *If $\mathcal{B}$ is invertible, then $\mathcal{B}_\pi$ is invertible.*
(v) *For the spectrum, we have $\sigma(\mathcal{B}_\pi) \subset \sigma(\mathcal{B})$.*

3. Spectral Components for $LH^2(\Omega^\# \times V)$

Let $X^\#$ be the dual space of X. For $\xi \in X^\#$, $x \in X$ we denote the pairing by $\langle \xi, x\rangle$. Then

$$\Omega^\# := \{\xi \in X^\# : \langle \xi, x\rangle > 0 \; \forall x \in \Omega\} \tag{3.1}$$

is called the (open) *dual cone* of Ω. If Ω is *homogeneous*, with simply transitive solvable group AN_Ω, one can show that the *adjoint action* $(g, \xi) \mapsto (g^t)^{-1}\xi$, for $g \in \mathrm{AN}_\Omega$ and $\xi \in X^\#$, is also simply transitive on $\Omega^\#$. Here we define

$$\langle g^t\xi, x\rangle := \langle \xi, gx\rangle \tag{3.2}$$

for all $\xi \in X^\#$, $x \in X$. For a *symmetric* cone Ω, realized as the open positive cone (1.2) of a Euclidean Jordan algebra X with unit element e, we may identify $X \approx X^\#$ via the scalar product $(x|y)$ on X, and (1.1) shows that $\Omega \approx \Omega^\#$ is *self-dual*. In this case (3.2) gives the usual transposition $g \mapsto g^t$, defined by

$$(g^t y|x) = (y|gx)$$

for all $x, y \in X$. We define $\varepsilon \in X^\#$ by $\langle \varepsilon, x\rangle := (e|x)$ for all $x \in X$. For each $\xi \in \Omega^\#$ there exists a unique $g_\xi \in \mathrm{AN}_\Omega$ such that

$$g_\xi^t \varepsilon = \xi.$$

Now let $D = T(\Omega, \Phi) \subset Z = U \times V$ be a symmetric Siegel domain, the special case of tube domains ($V = \{0\}$) being included.

PROPOSITION 3.1. *For fixed $\xi \in \Omega^\#$ consider the probability measure*

$$\mathrm{d}\gamma_\xi(v) = \Delta(\xi)^{d_2/r} \frac{\mathrm{d}v}{\pi^{d_2}} \mathrm{e}^{-\langle \xi, \Phi(v,v)\rangle} \tag{3.3}$$

on V. Then the Segal–Bargmann space

$$H^2_\xi(V) := \{\psi \in L^2(V, \mathrm{d}\gamma_\xi) : \psi \text{ holomorphic}\}$$

has the reproducing kernel

$$(v_1, v_2) \mapsto \mathrm{e}^{-\langle \xi, \Phi(v_1, v_2)\rangle}.$$

Proof. For $\xi = \varepsilon$ we have $\Delta(\varepsilon) = 1$ and

$$(v_1|v_2) := \langle \varepsilon, \Phi(v_1, v_2)\rangle$$

is the inner product on V. In this case the statement is classical. For general $\xi \in \Omega^\#$, write $\xi = g_\xi^t \varepsilon$ for a unique $g_\xi \in AN_\Omega$. Then we have

$$\begin{aligned}\langle \xi, \Phi(v_1, v_2)\rangle &= \langle g_\xi^t \varepsilon, \Phi(v_1, v_2)\rangle = \langle \varepsilon, g_\xi \Phi(v_1, v_2)\rangle = \langle \varepsilon, \Phi(g_\xi v_1, g_\xi v_2)\rangle \\ &= (g_\xi v_1 | g_\xi v_2).\end{aligned}$$

Since the transformation $v \mapsto u = g_\xi v$ satisfies $\mathrm{d}u = \Delta(\xi)^{d_2/r}\mathrm{d}v$, the assertion follows. □

Let $LH^2(\Omega^\# \times V)$ denote the Hilbert space of all measurable functions

$$\psi\colon\ \Omega^\# \times V \to \mathbb{C}, \quad (\xi, v) \mapsto \psi(\xi, v)$$

which are holomorphic in the second variable $v \in V$ and satisfy

$$\begin{aligned}\|\psi\|^2 &= \int_{\Omega^\#} \mathrm{d}\mu_{\Omega^\#}(\xi) \int_V \mathrm{d}\gamma_\xi(v) |\psi(\xi, v)|^2 \\ &= \int_{\Omega^\#} \mathrm{d}\mu_{\Omega^\#}(\xi)\Delta(\xi)^{d_2/r} \int_V \frac{\mathrm{d}v}{\pi^{d_2}}\, \mathrm{e}^{-\langle \xi, \Phi(v,v)\rangle} |\psi(\xi, v)|^2 < +\infty.\end{aligned}$$

Putting

$$(\pi(c)\psi)(\xi, v) := \psi(c^t \xi, c^{-1} v), \tag{3.4}$$

$$(\pi(t_{a,b})\psi)(\xi, v) := \mathrm{e}^{-\langle \xi, \Phi(v,b) - a + \Phi(b,b)/2\rangle} \psi(\xi, v + b) \tag{3.5}$$

for all $a \in iX,\ b \in V$ and $c \in GL(\Omega)$, it is easy to check that $\pi(c),\ \pi(t_{a,b})$ are unitary operators on $LH^2(\Omega^\# \times V)$, which satisfy

$$\begin{aligned}&\pi(c_1)\pi(c_2) = \pi(c_1 c_2), \\ &\pi(t_{a_1,b_1})\pi(t_{a_2,b_2}) = \pi(t_{a_1+a_2+(\Phi(b_2,b_1)-\Phi(b_1,b_2))/2,\ b_1+b_2}), \\ &\pi(c)\pi(t_{a,b}) = \pi(t_{ca,cb})\pi(c)\end{aligned}$$

for all $a, a_1, a_2 \in iX$, $b, b_1, b_2 \in V$ and $c, c_1, c_2 \in GL(\Omega)$. It follows that (3.4) and (3.5) define a unitary representation of $GL(\Omega) \triangleleft \Sigma$ on $LH^2(\Omega^\# \times V)$, given by

$$\pi(c t_{a,b})\psi(\xi, v) = \psi(c^t \xi, b + c^{-1} v)\mathrm{e}^{-\langle \xi, \Phi(v,cb) - ca + \Phi(cb,cb)/2\rangle}.$$

For any pair $\phi \in L^2(\Omega^\#)$, $u \in V$ define $K_{\phi,u} \in LH^2(\Omega^\# \times V)$ by

$$K_{\phi,u}(\xi, v) := \phi(\xi) \cdot \mathrm{e}^{\langle \xi, \Phi(v, g_\xi^{-1} u)\rangle}. \tag{3.6}$$

Then Proposition 3.1 implies for any $\psi \in LH^2(\Omega^\# \times V)$

$$\begin{aligned}(K_{\phi,u}|\psi) &= \int_{\Omega^\#} \mathrm{d}\mu_{\Omega^\#}(\xi)\Delta(\xi)^{d_2/r} \int_V \frac{\mathrm{d}v}{\pi^{d_2}} \mathrm{e}^{-\langle \xi, \Phi(v,v)\rangle} \overline{\phi(\xi)} \times \\ &\quad \times \mathrm{e}^{\langle \xi, \Phi(g_\xi^{-1} u, v)\rangle} \psi(\xi, v) \\ &= \int_{\Omega^\#} \mathrm{d}\mu_{\Omega^\#}(\xi) \overline{\phi(\xi)} \psi(\xi, g_\xi^{-1} u).\end{aligned} \tag{3.7}$$

It follows that the functions (3.6) form a total family in $LH^2(\Omega^\# \times V)$.

The AN-invariant measures $\mathrm{d}\mu_\Omega(x) = \mathrm{d}x\,\Delta(x)^{-d_1/r}$ and $\mathrm{d}\mu_{\Omega^\#}(\xi) = \mathrm{d}\xi\,\Delta(\xi)^{-d_1/r}$ on Ω and $\Omega^\#$, resp., are related by the *modulus function* on AN_Ω. More precisely [1] we have

$$\int_\Omega \mathrm{d}\mu_\Omega(x) f(x) = \int_{\Omega^\#} \mathrm{d}\mu_{\Omega^\#}(\xi)\Delta^*_{2\rho}(\xi) f(g_\xi e) \tag{3.8}$$

for all $f \in \mathcal{C}_c(\Omega)$, where $2\rho_k = \frac{a}{2}(2k - r - 1)$ for $1 \leqslant k \leqslant r$ and Δ^* denotes the conical function on $\Omega^\#$ (cf. (1.8)) for the reverse ordering $e_r, \ldots, e_1$ of the frame. We define the left Haar measure on AN_Ω by

$$\int_{\mathrm{AN}_\Omega} \mathrm{d}\lambda(c) f(ce) = \int_\Omega \mathrm{d}\mu_\Omega(x) f(x) \tag{3.9}$$

and use

$$\mathrm{d}\lambda(ct_{a,b}) = \mathrm{d}\lambda(c)\frac{\mathrm{d}a}{(2\pi)^{d_1}}\frac{\mathrm{d}b}{\pi^{d_2}}, \tag{3.10}$$

for $c \in \mathrm{AN}_\Omega$, $a \in iX$ and $b \in V$, as a left Haar measure on AN.

PROPOSITION 3.2. *For the representation space $LH^2(\Omega^\# \times V)$, the S-operator is a multiplication operator, given by*

$$(S\psi)(\xi, v) = \Delta^*_{2\rho - d/r}(\xi)\psi(\xi, v).$$

Proof. Let $\psi, \psi_1, \psi_2 \in LH^2(\Omega^\# \times V)$. We may assume that ψ_1 is of the form

$$\psi_1(\xi, v) = f_1(\xi)\mathrm{e}^{\langle \xi, \Phi(g_\xi^{-1}u_1, v)\rangle}$$

for $f_1 \in L^2(\Omega^\#)$, $u_1 \in V$. For any $\xi, \eta \in \Omega^\#$, $u \in V$ we have

$$\begin{aligned}
I(\xi) &:= \int_{\mathrm{AN}} \mathrm{d}\lambda(g)(\pi(g^{-1})\psi_2)(\xi, g_\xi^{-1}u_1)(\pi(g)\psi)(\eta, u)\\
&= \int_{\mathrm{AN}_\Omega} \mathrm{d}\lambda(c)\int_V \frac{\mathrm{d}b}{\pi^{d_2}}\int_{iX}\frac{\mathrm{d}a}{(2\pi)^{d_1}}(\pi(t_{-a,-b}c^{-1})\psi_2) \times\\
&\quad \times (\xi, g_\xi^{-1}u_1)(\pi(ct_{a,b})\psi)(\eta, u)\\
&= \int_{\mathrm{AN}_\Omega} \mathrm{d}\lambda(\gamma)\int_V \frac{\mathrm{d}b}{\pi^{d_2}}\int_{iX}\frac{\mathrm{d}a}{(2\pi)^{d_1}}(\pi(t_{-a,-b}\gamma^{-1}g_\eta)\psi_2) \times\\
&\quad \times (\xi, g_\xi^{-1}u_1)(\pi(g_\eta^{-1}\gamma t_{a,b})\psi)(\eta, u)
\end{aligned}$$

by making the change of variables $\gamma = g_\eta c$, $\mathrm{d}\lambda(\gamma) = \mathrm{d}\lambda(c)$. Since

$$\begin{aligned}
&(\pi(t_{-a,-b}\gamma^{-1}g_\eta)\psi_2)(\xi, g_\xi^{-1}u_1)\\
&\quad = \psi_2(g_\eta^t(\gamma^t)^{-1})\xi, g_\eta^{-1}\gamma(g_\xi^{-1}u_1 - b))\mathrm{e}^{-\langle \xi, a + \Phi(b,b)/2 - \Phi(g_\xi^{-1}u_1, b)\rangle}
\end{aligned}$$

and

$$(\pi(g_\eta^{-1}\gamma t_{a,b})\psi)(\eta,u) = \psi(\gamma^t\varepsilon, \gamma^{-1}g_\eta u + b)\mathrm{e}^{-\langle\gamma^t\varepsilon,\Phi(\gamma^{-1}g_\eta u,b)-a+\Phi(b,b)/2\rangle},$$

it follows that

$$\begin{aligned} I(\xi) &= \int_{\mathrm{AN}_\Omega} \mathrm{d}\lambda(\gamma)\int_V \frac{\mathrm{d}b}{\pi^{d_2}}\int_{iX}\frac{\mathrm{d}a}{(2\pi)^{d_1}}\psi_2(g_\eta^t(\gamma^t)^{-1}\xi, g_\eta^{-1}\gamma(g_\xi^{-1}u_1-b))\times \\ &\quad\times \mathrm{e}^{-\langle\xi,a+\Phi(b,b)/2-\Phi(g_\xi^{-1}u_1,b)\rangle}\psi(\gamma^t\varepsilon,\gamma^{-1}g_\eta u+b)\times \\ &\quad\times \mathrm{e}^{-\langle\gamma^t\varepsilon,\Phi(\gamma^{-1}g_\eta u,b)-a+\Phi(b,b)/2\rangle}. \end{aligned}$$

Using (3.8) and evaluating the a-integral by Fourier inversion we obtain, putting $\gamma = g_\vartheta$,

$$\begin{aligned} I(\xi) &= \int_{\Omega^\#}\mathrm{d}\mu_{\Omega^\#}(\vartheta)\Delta^*_{2\rho}(\vartheta)\int_V\frac{\mathrm{d}b}{\pi^{d_2}}\int_{iX}\frac{\mathrm{d}a}{(2\pi)^{d_1}}\psi_2(g_\eta^t(g_\vartheta^t)^{-1}\xi, g_\eta^{-1}g_\vartheta\times \\ &\quad\times(g_\xi^{-1}u_1-b))\mathrm{e}^{-\langle\xi,a+\Phi(b,b)/2-\Phi(g_\xi^{-1}u_1,b)\rangle}\psi(\vartheta,g_\vartheta^{-1}g_\eta u+b)\times \\ &\quad\times \mathrm{e}^{-\langle\vartheta,\Phi(g_\vartheta^{-1}g_\eta u,b)-a+\Phi(b,b)/2\rangle} \\ &= \Delta^*_{2\rho^*-d_1/r}(\xi)\int_V\frac{\mathrm{d}b}{\pi^{d_2}}\ \psi_2(\eta,g_\eta^{-1}(u_1-g_\xi b))\mathrm{e}^{-\langle\xi,\Phi(b,b)/2-\Phi(g_\xi^{-1}u_1,b)\rangle}\times \\ &\quad\times\psi(\xi,g_\xi^{-1}g_\eta u+b)\mathrm{e}^{-\langle\xi,\Phi(g_\xi^{-1}g_\eta u,b)+\Phi(b,b)/2\rangle} \\ &= \Delta^*_{2\rho-d/r}(\xi)\Delta(\xi)^{d_2/r}\int_V\frac{\mathrm{d}b}{\pi^{d_2}}\mathrm{e}^{-\langle\xi,\Phi(b,b)\rangle}\cdot\psi_2(\eta,g_\eta^{-1}(u_1-g_\xi b)\times \\ &\quad\times\psi(\xi,g_\xi^{-1}g_\eta u+b)\ \mathrm{e}^{\langle\xi,\Phi(g_\xi^{-1}(u_1-g_\eta u),b)\rangle} \\ &= \Delta^*_{2\rho-d/r}(\xi)\psi_2(\eta,u)\psi(\xi,g_\xi^{-1}u_1) \end{aligned}$$

by using the fact that $\psi_2(\eta, g_\eta^{-1}(u_1 - g_\xi b))\psi(\xi, g_\xi^{-1}g_\eta u + b)$ is holomorphic in b and therefore, by Proposition 3.1, the b-integral evaluates at $b = g_\xi^{-1}(u_1 - g_\eta u)$. On the other hand, (2.2) implies

$$\begin{aligned} &\psi_2(\eta,u)\int_{\Omega^\#}\mathrm{d}\mu_{\Omega^\#}(\xi)\overline{f_1(\xi)}(S\psi)(\xi,g_\xi^{-1}u_1) \\ &= \psi_2(\eta,u)(\psi_1|S\psi) = \int_{\mathrm{AN}}\mathrm{d}\lambda(g)(\pi(g)\psi_1|\psi_2)(\pi(g)\psi)(\eta,u) \\ &= \int_{\mathrm{AN}}\mathrm{d}\lambda(g)(\psi_1|\pi(g)^{-1}\psi_2)(\pi(g)\psi)(\eta,u) \\ &= \int_{\mathrm{AN}}\mathrm{d}\lambda(g)\int_{\Omega^\#}\mathrm{d}\mu_{\Omega^\#}(\xi)\overline{f_1(\xi)}(\pi(g)^{-1}\psi_2)(\xi,g_\xi^{-1}u_1)(\pi(g)\psi)(\eta,u) \\ &= \int_{\Omega^\#}\mathrm{d}\mu_{\Omega^\#}(\xi)\overline{f_1(\xi)}I(\xi). \end{aligned}$$

Therefore

$$\psi_2(\eta, u)(S\psi)(\xi, g_\xi^{-1}u_1) = I(\xi) = \Delta^*_{2\rho-d/r}(\xi)\psi_2(\eta, u)\psi(\xi, g_\xi^{-1}u_1)$$

and the assertion follows. □

THEOREM 3.3. *For the representation space $LH^2(\Omega^\# \times V)$, the spectral component $\mathcal{B}_\pi$ of an* AN*-invariant operator $\mathcal{B}$ has the kernel*

$$\begin{aligned}
&\mathcal{B}_\pi(\xi, v; \eta, u) \\
&= \Delta^*_{\rho-d/2r}(\xi)\Delta^*_{\rho-d/2r}(\eta)\mathrm{e}^{\langle\xi,\Phi(v,v)/2\rangle+\langle\eta,\Phi(u,u)/2\rangle}\times \\
&\times \int_{iX} \frac{\mathrm{d}\alpha}{(2\pi)^{d_1}}\mathcal{B}(g_\xi t_v, t_\alpha g_\eta t_u)\mathrm{e}^{\langle\alpha,\varepsilon\rangle}
\end{aligned}$$

for all $\xi, \eta \in \Omega^\#$ and $v, u \in V$.

Proof. Let $\psi \in LH^2(\Omega^\# \times V)$ and $\xi \in \Omega^\#$, $v \in V$. Applying the definition (2.4) and using left-invariance of $\mathrm{d}\lambda$ and $\mathcal{B}$ we obtain

$$\begin{aligned}
&(\mathcal{B}_\pi\psi)(\xi, v) \\
&= \int_{\mathrm{AN}_\Omega} \mathrm{d}\lambda(c)\int_V \frac{\mathrm{d}b}{\pi^{d_2}}\int_{iX}\frac{\mathrm{d}a}{(2\pi)^{d_1}}\mathcal{B}(\mathbf{1}, ct_{a,b})(S^{1/2}\pi(ct_{a,b})S^{-1/2}\psi)(\xi, v) \\
&= \int_{\mathrm{AN}_\Omega} \mathrm{d}\lambda(\gamma)\int_V \frac{\mathrm{d}b}{\pi^{d_2}}\int_{iX}\frac{\mathrm{d}a}{(2\pi)^{d_1}}\mathcal{B}(\mathbf{1}, g_\xi^{-1}\gamma t_{a,b})\times \\
&\times (S^{1/2}\pi(g_\xi^{-1}\gamma t_{a,b})S^{-1/2}\psi)(\xi, v)\times \\
&\times \int_{\mathrm{AN}_\Omega} \mathrm{d}\lambda(\gamma)\int_V \frac{\mathrm{d}b}{\pi^{d_2}}\int_{iX}\frac{\mathrm{d}a}{(2\pi)^{d_1}}\mathcal{B}(g_\xi t_v, t_{g_\xi v}t_{\gamma a,\gamma b}\gamma)\times \\
&\times (S^{1/2}\pi(g_\xi^{-1}t_{\gamma a,\gamma b}\gamma)S^{-1/2}\psi)(\xi, v) \\
&= \int_{\mathrm{AN}_\Omega} \mathrm{d}\lambda(\gamma)\Delta(\gamma^t\varepsilon)^{-d_1/r}\int_V \frac{\mathrm{d}\beta}{\pi^{d_2}}\int_{iX}\frac{\mathrm{d}\alpha}{(2\pi)^{d_1}}\mathcal{B}(g_\xi t_v, t_\alpha\gamma t_\beta)\times \\
&\times (S^{1/2}\pi(t_{-v}g_\xi^{-1}t_\alpha\gamma t_\beta)S^{-1/2}\psi)(\xi, v).
\end{aligned} \tag{3.11}$$

Here we made the successive change of variables

$$\begin{aligned}
&\gamma := g_\xi c, \qquad \mathrm{d}\lambda(\gamma) = \mathrm{d}\lambda(c), \\
&\beta := b + \gamma^{-1}g_\xi v, \qquad \mathrm{d}\beta = \mathrm{d}b, \\
&\alpha := \gamma a + (\Phi(\gamma\beta, g_\xi v) - \Phi(g_\xi v, \gamma\beta))/2, \\
&\mathrm{d}\alpha = \Delta(\gamma e)^{d_1/r}\mathrm{d}a = \Delta(\gamma^t\varepsilon)^{d_1/r}\mathrm{d}a.
\end{aligned}$$

Then (1.3) and (1.6) imply $t_{g_\xi v}t_{\gamma a,\gamma b}\gamma = t_{\gamma a+(\Phi(\gamma b,g_\xi v)-\Phi(g_\xi v,\gamma b))/2, g_\xi v+\gamma b}\gamma = t_{\alpha,\gamma\beta}\gamma = t_\alpha t_{\gamma\beta}\gamma = t_\alpha\gamma t_\beta$ and hence $g_\xi^{-1}\, t_{\gamma a,\gamma b}\gamma = g_\xi^{-1}t_{g_\xi v}^{-1}t_\alpha\gamma t_\beta = (t_{g_\xi v}g_\xi)^{-1}\times$

$\times t_\alpha \gamma t_\beta = (g_\xi t_v)^{-1} t_\alpha \gamma t_\beta = t_{-v} g_\xi^{-1} t_\alpha \gamma t_\beta$. Using Proposition 3.2 and applying (3.4), (3.5) repeatedly, we obtain

$$\begin{aligned}
&\Delta^*_{d/2r-\rho}(\xi) S^{1/2}(\pi(t_{-v} g_\xi^{-1} t_\alpha \gamma t_\beta) S^{-1/2}\psi)(\xi, v)\\
&\quad = (\pi(t_{-v} g_\xi^{-1} t_\alpha \gamma t_\beta) S^{-1/2}\psi)(\xi, v)\\
&\quad = (\pi(g_\xi^{-1} t_\alpha \gamma t_\beta) S^{-1/2}\psi)(\xi, 0)\mathrm{e}^{\langle \xi, \Phi(v,v)/2\rangle}\\
&\quad = (\pi(t_\alpha \gamma t_\beta) S^{-1/2}\psi)(\varepsilon, 0)\mathrm{e}^{\langle \xi, \Phi(v,v)/2\rangle}\\
&\quad = (\pi(\gamma t_\beta) S^{-1/2}\psi)(\varepsilon, 0)\mathrm{e}^{\langle \varepsilon, \alpha\rangle}\mathrm{e}^{\langle \xi, \Phi(v,v)/2\rangle}\\
&\quad = (\pi(t_\beta) S^{-1/2}\psi)(\gamma^t\varepsilon, 0)\mathrm{e}^{\langle \varepsilon, \alpha\rangle}\mathrm{e}^{\langle \xi, \Phi(v,v)/2\rangle}\\
&\quad = \Delta^*_{d/2r-\rho}(\gamma^t\varepsilon)\psi(\gamma^t\varepsilon, \beta)\mathrm{e}^{\langle \varepsilon, \alpha\rangle}\mathrm{e}^{\langle \xi, \Phi(v,v)/2\rangle}\mathrm{e}^{-\langle \gamma^t\varepsilon, \Phi(\beta,\beta)/2\rangle}.
\end{aligned}$$

Now combine (3.11), (3.12) and use (3.6), putting $\gamma = g_\eta$. Then

$$\begin{aligned}
&(\mathcal{B}_\pi \psi)(\xi, v)\\
&\quad = \int_{\mathrm{AN}_\Omega} \mathrm{d}\lambda(\gamma)\Delta(\gamma^t\varepsilon)^{-d_1/r}\int_V \frac{\mathrm{d}\beta}{\pi^{d_2}}\int_{iX}\frac{\mathrm{d}\alpha}{(2\pi)^{d_1}}\mathcal{B}(g_\xi t_v,\ t_\alpha\gamma t_\beta)\times\\
&\qquad \times \Delta^*_{\rho-d/2r}(\xi)\Delta^*_{d/2r-\rho}(\gamma^t\varepsilon)\psi(\gamma^t\varepsilon, \beta)\mathrm{e}^{\langle \varepsilon, \alpha\rangle}\mathrm{e}^{\langle \xi, \Phi(v,v)/2\rangle}\mathrm{e}^{-\langle \gamma^t\varepsilon, \Phi(\beta,\beta)/2\rangle}\\
&\quad = \int_{\Omega^\#}\mathrm{d}\mu_{\Omega^\#}(\eta)\Delta^*_{2\rho}(\eta)\Delta(\eta)^{-d_1/r}\int_V\frac{\mathrm{d}\beta}{\pi^{d_2}}\int_{iX}\frac{\mathrm{d}\alpha}{(2\pi)^{d_1}}\mathcal{B}(g_\xi t_v, t_\alpha g_\eta t_\beta)\times\\
&\qquad \times \Delta^*_{\rho-d/2r}(\xi)\Delta^*_{d/2r-\rho}(\eta)\psi(\eta, \beta)\mathrm{e}^{\langle \varepsilon, \alpha\rangle}\mathrm{e}^{\langle \xi, \Phi(v,v)/2\rangle}\mathrm{e}^{-\langle \eta, \Phi(\beta,\beta)/2\rangle}\\
&\quad = \Delta^*_{\rho-d/2r}(\xi)\mathrm{e}^{\langle \xi, \Phi(v,v)/2\rangle}\int_{\Omega^\#}\mathrm{d}\mu_{\Omega^\#}(\eta)\Delta^*_{\rho-d/2r}(\eta)\Delta(\eta)^{d_2/r}\times\\
&\qquad \times \int_V\frac{\mathrm{d}\beta}{\pi^{d_2}}\mathrm{e}^{-\langle \eta, \Phi(\beta,\beta)\rangle}\psi(\eta, \beta)\mathrm{e}^{\langle \eta, \Phi(\beta,\beta)/2\rangle}\times\\
&\qquad \times \int_{iX}\frac{\mathrm{d}\alpha}{(2\pi)^{d_1}}\mathcal{B}(g_\xi t_v, t_\alpha g_\eta t_\beta)\mathrm{e}^{\langle \varepsilon, \alpha\rangle}.
\end{aligned}$$

□

4. Berezin Transform for Gindikin–Bergman Spaces

An important example of a G-invariant operator, for a Lie group G acting holomorphically on a complex domain D, is the *Berezin transformation* [2, 6] associated with a Hilbert space H of holomorphic functions on D which has a reproducing kernel and carries an irreducible unitary representation of G.

For the symmetric Siegel domain $D = T(\Omega, \Phi)$ (including the tube type domains $T(\Omega)$) and $G := \mathrm{AN}$, the relevant Hilbert spaces are the *Gindikin–Bergman spaces*

$$H^2_\nu(D) := \{h \in L^2(D, \mathrm{d}\mu_\nu) : h \text{ holomorphic}\} \tag{4.1}$$

with respect to the measure

$$d\mu_\nu(z) = \frac{dz}{\pi^d} \frac{\Delta_{\nu-2d_1/r-d_2/r}(\tau(z,z))}{\Gamma_\Omega(\nu^*)\Gamma_\Omega(\nu-d/r)}$$

on D. Here $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$ is a multi-parameter satisfying $\nu_i > d/r + (i-1)a/2$ for all i, and we put

$$\nu^* := (\nu_r, \ldots, \nu_1).$$

For multi-parameters ν, β and $z_1, z_2 \in D$ one can show [4] that

$$\int_D \frac{dz}{\pi^d} - \frac{\Gamma_\Omega(\nu^*)}{\Delta_\nu(\tau(z_1,z))} \frac{\Delta_{\beta^*}(\tau(z,z))}{\Gamma_\Omega(\beta^* + d_1/r)} \frac{\Gamma_\Omega(\nu^*)}{\Delta_\nu(\tau(z,z_2))} = \frac{\Gamma_\Omega(2\nu^* - \beta - 2d_1/r - d_2/r)}{\Delta_{2\nu-\beta^*-2d_1/r-d_2/r}(\tau(z_1,z_2))}.$$

Putting $\beta = \nu^* - 2d_1/r - d_2/r$ this implies

$$\int_D \frac{dz}{\pi^d} \frac{\Gamma_\Omega(\nu^*)}{\Delta_\nu(\tau(z_1,z))} \frac{\Delta_{\nu-2d_1/r-d_2/r}(\tau(z,z))}{\Gamma_\Omega(\nu-d/r)} \frac{\Gamma_\Omega(\nu^*)}{\Delta_\nu(\tau(z,z_2))} = \frac{\Gamma_\Omega(\nu^*)}{\Delta_\nu(\tau(z_1,z_2))}.$$

This shows that $H^2_\nu(D)$ has the reproducing kernel

$$K_\nu(z,w) = \Delta_{-\nu}(\tau(z,w)), \quad z, w \in D. \tag{4.2}$$

Putting $K^\nu_w(z) := K_\nu(z,w)$, we obtain the *Berezin transform*

$$\begin{aligned} (\mathcal{B}^\nu f)(z) &= \frac{(K^\nu_z | f K^\nu_z)}{K_\nu(z,z)} = \int_D d\mu_\nu(w) \frac{K_\nu(z,w) f(w) K_\nu(w,z)}{K_\nu(z,z)} \\ &= \frac{\Gamma_\Omega(\nu^*)}{\Gamma_\Omega(\nu - \frac{d}{r})} \int_D \frac{dw}{\pi^d} \Delta_{-2d_1/r-d_2/r}(\tau(w,w)) \times \\ &\quad \times \frac{\Delta_\nu(\tau(z,z))\Delta_\nu(\tau(w,w))}{\Delta_\nu(\tau(z,w))\Delta_\nu(\tau(w,z))}. \end{aligned}$$

This shows that $\mathcal{B}^\nu$, as an integral operator on $D \approx$ AN with respect to the Haar measure (3.10), has the integral kernel

$$\mathcal{B}^\nu(z,w) = \frac{\Gamma_\Omega(\boldsymbol{\nu}^*)}{\Gamma_\Omega(\boldsymbol{\nu} - \frac{d}{r})} \frac{\Delta_\nu(\tau(z,z))\Delta_\nu(\tau(w,w))}{\Delta_\nu(\tau(z,w))\Delta_\nu(\tau(w,z))} \tag{4.3}$$

for all $z, w \in D$.

Now consider the AN-representation space $LH^2(\Omega^\# \times V)$ introduced in Section 3, and let $\mathcal{B}^\nu_\pi$ denote the associated spectral component of the Berezin transform $\mathcal{B}^\nu$. As our main result, we will express the integral kernel of $\mathcal{B}^\nu_\pi$ in closed form, using the theory of multivariable special functions, more precisely the generalized hypergeometric functions associated with symmetric (or homogeneous) cones [4].

In the multiparameter case the representation of hypergeometric functions as generalized Euler integrals is more convenient than the approach via hypergeometric series. Accordingly, our starting point is the hypergeometric function

$$
\begin{aligned}
&{}_{\Omega^\#}F(\rho,\sigma,\tau)(\lambda) \\
&= \frac{\Gamma_{\Omega^\#}(\tau)}{\Gamma_{\Omega^\#}(\sigma)\Gamma_{\Omega^\#}(\tau-\sigma)} \times \\
&\times \int_{\Omega^\# \cap (\varepsilon - \Omega^\#)} d\mu_{\Omega^\#}(\xi)\Delta^*_\sigma(\xi)\Delta^*_{\tau-\sigma-d/r}(\varepsilon-\xi)\Delta^*_{-\rho}(\varepsilon - g_\xi\lambda)
\end{aligned} \tag{4.4}
$$

introduced in [4, Definition 4.1]. Here $\lambda \in \Omega^\# \cap (\varepsilon - \Omega^\#)$ or $\lambda \in -\Omega^\#$. In the second case there is a more convenient representation

$$
\begin{aligned}
&{}_{\Omega^\#}F(\rho,\sigma,\tau)(-\lambda) \\
&= \frac{\Gamma_{\Omega^\#}(\tau)}{\Gamma_{\Omega^\#}(\sigma)\Gamma_{\Omega^\#}(\tau-\sigma)} \frac{1}{\Delta^*_{\tau-d/r}(\lambda)} \times \\
&\times \int_{\Omega^\# \cap (\lambda - \Omega^\#)} d\mu_{\Omega^\#}(\xi)\Delta^*_\sigma(\xi)\Delta^*_{-\rho}(\varepsilon+\xi)\Delta^*_{\tau-\sigma-d/r}(\lambda-\xi)
\end{aligned} \tag{4.5}
$$

valid for $\lambda \in \Omega^\#$ [4, (4.12)]. The functions (4.4), (4.5) are multi-variable generalizations of the classical ${}_2F_1$-hypergeometric functions. For our purposes, we need a limiting case analogous to the classical Ψ-function [5], which in turn is closely related to the Bessel functions K_ν and the confluent hypergeometric function ${}_1F_1$. Accordingly, we define for any fixed $x \in \Omega$

$$
\Psi_\Omega(\rho,\sigma)(x) = \lim_{\tau\to+\infty} {}_\#F(\rho,\sigma,\tau)(\varepsilon - \tau(g_x^t)^{-1}\varepsilon), \tag{4.6}
$$

where $\tau > 0$ is a scalar, identified with $(\tau,\dots,\tau)$, and $g_x \in \mathrm{AN}_\Omega$ satisfied $g_x e = x$. Note that $\varepsilon - \tau(g_x^t)^{-1}\varepsilon \in -\Omega^\#$ if τ is sufficiently large. Hence (4.5) implies

$$
\begin{aligned}
&{}_{\Omega^\#}F(\rho,\sigma,\tau)(\varepsilon - \tau(g_x^t)^{-1}\varepsilon) \\
&= \frac{1}{\Gamma_{\Omega^\#}(\sigma)} \int_{\Omega^\# \cap (\tau(g_x^t)^{-1}\varepsilon - \varepsilon - \Omega^\#)} d\mu_{\Omega^\#}(\xi)\Delta^*_\sigma(\xi)\Delta^*_{-\rho}(\varepsilon+\xi) f_\tau(\xi),
\end{aligned} \tag{4.7}
$$

where

$$
f_\tau(\xi) := \frac{\Gamma_{\Omega^\#}(\tau)}{\Gamma_{\Omega^\#}(\tau-\sigma)} \frac{\Delta^*_{\tau-\sigma-d/r}(\tau(g_x^t)^{-1}\varepsilon - \varepsilon - \xi)}{\Delta^*_{\tau-d/r}(\tau(g_x^t)^{-1}\varepsilon - \varepsilon)}.
$$

Since $g_x^t \in \mathrm{AN}_{\Omega^\#}$ we have

$$
\Delta^*_{\tau-d/r}(\tau(g_x^t)^{-1}\varepsilon - \varepsilon) = \Delta^*_{\tau-d/r}(\tau(g_x^t)^{-1}\varepsilon)\Delta^*_{\tau-d/r}\left(\varepsilon - g_x^t\frac{\varepsilon}{\tau}\right)
$$

and, similarly,

$$
\begin{aligned}
&\Delta^*_{\tau-\sigma-d/r}(\tau(g_x^t)^{-1}\varepsilon - \varepsilon - \xi) \\
&= \Delta^*_{\tau-\sigma-d/r}(\tau(g_x^t)^{-1}\varepsilon)\Delta^*_{\tau-\sigma-d/r}\left(\varepsilon - g_x^t\frac{\varepsilon+\xi}{\tau}\right).
\end{aligned}
$$

For $\tau \to +\infty$ it follows that

$$\frac{\Delta^*_{\tau-\sigma-d/r}(\varepsilon - g_x^t \frac{\varepsilon+\xi}{\tau})}{\Delta^*_{\tau-d/r}(\varepsilon - g_x^t \frac{\varepsilon}{\tau})} \longrightarrow \frac{e^{-\langle g_x^t(\varepsilon+\xi), e\rangle}}{e^{-\langle g_x^t \varepsilon, e\rangle}} = e^{-\langle g_x^t \xi, e\rangle} = e^{-\langle \xi, g_x e\rangle} = e^{-\langle \xi, x\rangle}.$$

On the other hand

$$\frac{\Delta^*_{\tau-\sigma-d/r}(\tau (g_x^t)^{-1}\varepsilon)}{\Delta^*_{\tau-d/r}(\tau (g_x^t)^{-1}\varepsilon)} = \Delta^*_{-\sigma}(\tau (g_x^t)^{-1}\varepsilon) = \tau^{-|\sigma|}\Delta_{\sigma^*}(x).$$

Since $\Gamma_{\Omega^\#}(\tau)/\Gamma_{\Omega_\#}(\tau - \sigma)\tau^{|\sigma|} \to 1$ as $\tau \to +\infty$, it follows that $f_\tau(\xi) \to \Delta_{\sigma^*}(x)e^{-\langle \xi, x\rangle}$ pointwise on $\Omega^\#$. By Lebesgue theory we obtain from (4.7)

$$\Psi_\Omega(\rho, \sigma)(x) = \Delta_{\sigma^*}(x) \int_{\Omega^\#} d\mu_{\Omega^\#}(\xi) \Delta^*_\sigma(\xi) \Delta^*_{-\rho}(\varepsilon + \xi) e^{-\langle \xi, x\rangle}. \tag{4.8}$$

THEOREM 4.1. *The Berezin transform $\mathcal{B}^\nu$ associated with $H^2_\nu(D)$ has the spectral component $\mathcal{B}^\nu_\pi$, for the representation space $LH^2(\Omega^\# \times V)$, with integral kernel*

$$\begin{aligned}\mathcal{B}^\nu_\pi(\xi, v; \eta, u) \;=\; & 4^{r\nu} \frac{\Delta^*_{\rho+\nu^*-d/2r}(\xi)\Delta^*_{\rho+\nu^*-d/2r}(\eta)}{\Gamma_\Omega(\nu^*)\Gamma_\Omega(\nu - d/r)} e^{-\langle \xi+\eta, e\rangle} e^{(g_\xi v | g_\eta u)} \times \\ & \times \Delta_{-\nu}(2g_\xi e + 2g_\eta e + \Phi(g_\xi v - g_\eta u, g_\xi v - g_\eta u)) \times \\ & \times \Psi_\Omega(d_1/r - \nu^*, \nu^*) \times \\ & \times (2g_\xi e + 2g_\eta e + \Phi(g_\xi v - g_\eta u, g_\xi v - g_\eta u))\end{aligned}$$

for all $\xi, \eta \in \Omega^\#$ and $u, v \in V$, where Ψ_Ω is the Ψ-function of the cone Ω, defined in (4.6) *and given explicitly by* (4.8).

Proof. Given a function σ on $\overline{\Omega}^\#$, let

$$\tilde{\sigma}(s) := \int_{\Omega^\#} dt\, e^{-\langle t, s\rangle} \sigma(t), \quad s \in T(\Omega)$$

denote the Laplace transform, considered as a holomorphic function on $T(\Omega)$. Let

$$\hat{f}(t) := \int_{iX} \frac{da}{(2\pi)^{d_1}} e^{-\langle t, a\rangle} f(a)$$

denote the Fourier transform of a function f on iX. Then, for fixed $s \in T(\Omega)$, the function $f_s(a) := \tilde{\sigma}(s - a)$ satisfies

$$\hat{f}_s(t) = e^{-\langle t, a\rangle} \sigma(t)$$

for all $t \in \overline{\Omega}^\#$, as follows by Fourier inversion. Similarly, the function $h(a) := e^{-\langle \varepsilon, a\rangle} f_s(a)$ satisfies

$$\hat{h}(t) = \hat{f}_s(t + \varepsilon) = e^{-\langle t+\varepsilon, s\rangle} \sigma(t + \varepsilon)$$

for all $t \in \overline{\Omega}^{\#}$. Applying Parseval's formula, it follows that, for fixed $s \in T(\Omega)$, we have

$$\int_{iX} \frac{\mathrm{d}a}{(2\pi)^{d_1}} \mathrm{e}^{-\langle\varepsilon,a\rangle} \overline{\tilde{\sigma}_1(s-a)} \tilde{\sigma}_2(s-a) = \mathrm{e}^{-\langle\varepsilon,s\rangle} \int_{\overline{\Omega}^{\#}} \mathrm{d}t \overline{\sigma_1(t)} \sigma_2(t+\varepsilon) \mathrm{e}^{-\langle t,s+s^*\rangle} \tag{4.9}$$

if σ_1, σ_2 are functions on $\overline{\Omega}^{\#}$, with Laplace transforms $\tilde{\sigma}_1, \tilde{\sigma}_2$, resp. Now let $\mathcal{B}$: $D \times D \to \mathbb{C}$ be an AN-invariant kernel of the form

$$\mathcal{B}(z,w) = K_1(\tau(z,z)) K_2(\tau(w,w)) \overline{\tilde{\sigma}_1(\tau(z,w))} \tilde{\sigma}_2(\tau(z,w)), \tag{4.10}$$

where K_1, K_2: $\Omega \to \mathbb{R}$ and σ_1, σ_2 are functions on $\overline{\Omega}^{\#}$, with Laplace transforms $\tilde{\sigma}_1, \tilde{\sigma}_2$ resp. Since $g_\xi t_v(e) = (g_\xi e + \Phi(g_\xi v, g_\xi v)/2, g_\xi v)$, we have $\tau(g_\xi t_v(e), g_\xi t_v(e)) = 2g_\xi e$ and

$$\begin{aligned} s &:= \tau(g_\xi t_v(e), g_\eta t_u(e)) \\ &= g_\xi e + g_\eta e + \frac{\Phi(g_\xi v, g_\xi v)}{2} + \frac{\Phi(g_\eta u, g_\eta u)}{2} - \Phi(g_\xi v, g_\eta u). \end{aligned}$$

This implies $\langle\varepsilon, s\rangle = \langle\xi+\eta, e\rangle + \langle\xi, \Phi(v,v)/2\rangle + \langle\eta, \Phi(u,u)/2\rangle - (g_\xi v|g_\eta u)$ and $s + s^* = 2g_\xi e + 2g_\eta e + \Phi(g_\xi v - g_\eta u, g_\xi v - g_\eta u)$. Since $\tau(z, t_a w) = \tau(z,w) - a$ for all $a \in iX$, (4.9) yields

$$\begin{aligned} &\int_{iX} \frac{\mathrm{d}a}{(2\pi)^{d_1}} \mathcal{B}(g_\xi t_v(e), t_a g_\eta t_u(e)) \mathrm{e}^{-\langle\varepsilon,a\rangle} \\ &= K_1(2g_\xi e) K_2(2g_\eta e) \int_{iX} \frac{\mathrm{d}a}{(2\pi)^{d_1}} \mathrm{e}^{-\langle\varepsilon,a\rangle} \overline{\tilde{\sigma}_1(s-a)} \tilde{\sigma}_2(s-a) \\ &= K_1(2g_\xi e) K_2(2g_\eta e) \mathrm{e}^{-\langle\xi+\eta,e\rangle} \mathrm{e}^{-\langle\xi,\Phi(v,v)/2> - \langle\eta,\Phi(u,u)/2\rangle} \mathrm{e}^{(g_\xi v|g_\eta u)} \times \\ &\quad \times \int_{\overline{\Omega}^{\#}} \mathrm{d}t \overline{\sigma_1(t)} \sigma_2(t+\varepsilon) \mathrm{e}^{-\langle t, 2g_\xi e + 2g_\eta e + \Phi(g_\xi v - g_\eta u, g_\xi v - g_\eta u)\rangle}. \end{aligned}$$

Applying Theorem 3.3, we see that $\mathcal{B}_\pi$ has the integral kernel

$$\begin{aligned} &\mathcal{B}_\pi(\xi, v; \eta, u) \\ &= K_1(2g_\xi e) K_2(2g_\eta e) \mathrm{e}^{-\langle\xi+\eta,e\rangle} \mathrm{e}^{(g_\xi v|g_\eta u)} \Delta^*_{\rho-d/2r}(\xi) \Delta^*_{\rho-d/2r}(\eta) \times \\ &\quad \times \int_{\overline{\Omega}^{\#}} \mathrm{d}t \overline{\sigma_1(t)} \sigma_2(t+\varepsilon) \mathrm{e}^{-\langle t, 2g_\xi e + 2g_\eta e + \Phi(g_\xi v - g_\eta u, g_\xi v - g_\eta u)\rangle}. \end{aligned} \tag{4.11}$$

By [3, Corollary VII.1.3] we have

$$\Delta_{-\nu}(s) = \frac{1}{\Gamma_\Omega(\nu^*)} \int_{\Omega^{\#}} \mathrm{d}t \mathrm{e}^{-\langle t,s\rangle} \Delta_{\nu^*-d_1/r}(t) = \frac{1}{\Gamma_\Omega(\nu^*)} \tilde{\Delta}_{\nu^*-d_1/r}(s)$$

for all $s \in T(\Omega)$. It follows that the integral kernel (4.3) of $\mathcal{B}^\nu$ has the form

$$\mathcal{B}^\nu(z,w) = \frac{\Delta_\nu(\tau(z,z)) \Delta_\nu(\tau(w,w))}{\Gamma_\Omega(\nu^*) \Gamma_\Omega(\nu - d/r)} \tilde{\Delta}_{\nu^*-d_1/r}(\tau(z,w)) \tilde{\Delta}_{\nu^*-d_1/r}(\tau(w,z)).$$

Taking $K_1 := \Delta_\nu/\Gamma_\Omega(\nu^*)$, $K_2 := \Delta_\nu/\Gamma_\Omega(\nu - d/r)$, $\sigma_1 = \sigma_2 := \Delta^*_{\nu^*-d_1/r}$ in (4.10) and putting

$$s := 2g_\xi e + 2g_\eta e + \Phi(g_\xi v - g_\eta u, g_\xi v - g_\eta u) \in \Omega,$$

(4.11) and (4.8) imply

$$\begin{aligned}
&\mathcal{B}^\nu_\pi(\xi, v; \eta, u)\\
&= \frac{\Delta_\nu(2g_\xi e)\Delta_\nu(2g_\eta e)}{\Gamma_\Omega(\nu^*)\Gamma_\Omega(\nu - \frac{d}{r})} e^{-\langle \xi+\eta, e\rangle} e^{(g_\xi v|g_\eta u)} \times\\
&\times \Delta^*_{\rho-d/2r}(\xi)\Delta^*_{\rho-d/2r}(\eta) \int_{\Omega^\#} dt\, \Delta^*_{\nu^*-d_1/r}(t)\Delta^*_{\nu^*-d_1/r}(t+\varepsilon) e^{-\langle t,s\rangle}\\
&= \frac{4^{r\nu}}{\Gamma_\Omega(\nu^*)\Gamma_\Omega(\nu - \frac{d}{r})} \Delta^*_{\rho+\nu^*-d/2r}(\xi)\Delta^*_{\rho+\nu^*-d/2r}(\eta) e^{-\langle \xi+\eta, e\rangle} \times\\
&\times e^{(g_\xi v|g_\eta u)} \Delta_{-\nu}(s)\Psi(d_1/r - \nu^*, \nu^*)(s). \qquad \square
\end{aligned}$$

5. Correspondence Principle for Berezin Transforms

In this section we prove that the correspondence principle holds for the Berezin transform $\mathcal{B}^{\boldsymbol{\nu}}$ on $D = T(\Omega, \Phi)$, namely $\mathcal{B}^{\boldsymbol{\nu}} \to I$, strongly as $\boldsymbol{\nu} \to \infty$. We shall reduce this to general summability arguments.

Consider a topological group G with left Haar measure $d\lambda(g)$.

DEFINITION 5.1. A net $\{F_\alpha\}_{\alpha\in J}$ (where J is a directed set) of functions in $L^1(G, d\lambda)$ is called a *summability kernel* if

(i) $\int_G d\lambda(h) F_\alpha(h) = 1,\ \forall \alpha \in J$.
(ii) $C := \sup_{\alpha\in J} \int_G d\lambda(h)|F_\alpha(h)| < \infty$.
(iii) There is a family $\{U_\delta\}_{0<\delta<1}$ of open neighborhoods of the identity element $\mathbf{1} \in G$, so that $\bigcap_{\delta\in(0,1)} U_\delta = \{\mathbf{1}\}$, $U_{\delta_1} \subseteq U_{\delta_2}$ if $0 < \delta_1 < \delta_2 < 1$ and

$$\lim_\alpha \int_{G\setminus U_\delta} d\lambda(h)|F_\alpha(h)| = 0, \quad \forall \delta \in (0, 1). \tag{5.1}$$

If $F_\alpha(g) \geqslant 0$ almost everywhere on G for all $\alpha \in J$ then the kernel $\{F_\alpha\}_{\alpha\in J}$ is called *positive*. In this case (ii) above reduces to (i), with $C = 1$.

The *right translation* operator on functions on G is defined by

$$(R_h f)(g) = f(gh). \tag{5.2}$$

DEFINITION 5.2. A Banach space X of measurable functions on G is called *right homogeneous* if

(a) For all $f \in X$ and $h \in G$, $R_h f \in X$ and $\|R_h f\|_X = \|f\|_X$ (i.e. G acts on X isometrically by right translations).
(b) For every $f \in X$ the map $G \ni h \mapsto R_h f \in X$ is continuous in the norm of X.
(c) Norm-convergence in X implies almost everwhere convergence.

For every $F \in L^1(G, \mathrm{d}\lambda)$ and $f \in X$ the integral

$$F \circledast f := \int_G \mathrm{d}\lambda(h) F(h) R_h(f) \tag{5.3}$$

converges absolutely in the norm of X (by (a) and (b)), i.e. it is a Bochner integral. In view of (c) it converges almost everywhere on G and

$$(F \circledast f)(g) := \int_G \mathrm{d}\lambda(h) F(h) f(gh) = \int_G \mathrm{d}\lambda(h) F(g^{-1}h) f(h) \tag{5.4}$$

almost everywhere in G. Thus $F \circledast f$ is essentially the convolution of F and f. If $\mathcal{B}$ is a left-invariant operator on G with kernel $\mathcal{B}(g, h)$, i.e.

$$(\mathcal{B} f)(g) = \int_G \mathrm{d}\lambda(h) \mathcal{B}(g, h) f(h) \tag{5.5}$$

and

$$\mathcal{B}(g_1 g, g_1 h) \quad \forall g_1, g, g \in G, \tag{5.6}$$

then (5.4) shows that

$$\mathcal{B} f = F \circledast f, \tag{5.7}$$

where

$$F(h) := \mathcal{B}(\mathbf{1}, h). \tag{5.8}$$

Conversely, given $F \in L^1(G, \mathrm{d}\lambda)$, define

$$\mathcal{B}(g, h) := F(g^{-1}h), \quad \forall g, h \in G. \tag{5.9}$$

Then (5.6), (5.7) and (5.8) hold.

THEOREM 5.3. *Let $\{F_\alpha\}_{\alpha \in J}$ be a summability kernel on G, and let X be a right-homogeneous Banach space on G. Then*

$$\lim_\alpha \|F_\alpha \circledast f - f\|_X = 0, \quad \forall f \in X. \tag{5.10}$$

Consequently,

$$\lim_\alpha (F_\alpha \circledast f)(g) = \lim_\alpha \int_G \mathrm{d}\lambda(h) F(h) f(gh) = f(g) \tag{5.11}$$

almost everywhere on G.

Proof. Choose $\{U_\delta\}_{\delta\in(0,1)}$ as in Definition 5.1(iii). For every $f \in X$ define its *modulus of continuity* by

$$w_{f,X}(\delta) = \sup_{h\in U_\delta} \|R_h(f) - f\|_X, \quad \delta \in (0,1).$$

The continuity of the right action of G on X implies that

$$\lim_{\delta\to 0} w_{f,X}(\delta) = 0, \quad \forall f \in X.$$

Fix $f \in X$ and $\varepsilon > 0$. Choose $\delta \in (0,1)$ such that $w_{f,X}(\delta) < \varepsilon$. Then, by Definition 5.1(i),

$$F_\alpha \circledast f - f = \int_G \mathrm{d}\lambda(h) F(h)(R_h f - f), \quad \forall \alpha \in J.$$

It follows that

$$\begin{aligned}
\|F_\alpha \circledast f - f\|_X &\leqslant \int_G \mathrm{d}\lambda(h)|F_\alpha(h)|\|R_h f - f\|_X \\
&= \int_{U_\delta} \mathrm{d}\lambda(h)|F_\alpha(h)|\|R_h f - f\|_X + \\
&\quad + \int_{G\setminus U_\delta} \mathrm{d}\lambda(h)|F_\alpha(h)|\|R_h f - f\|_X \\
&\leqslant w_{f,X}(\delta)\int_{U_\delta} \mathrm{d}\lambda(h)|F_\alpha(h)| + 2\|f\|_X \int_{G\setminus U_\delta} \mathrm{d}\lambda(h)|F_\alpha(h)| \\
&\leqslant \varepsilon C + 2\|f\|_X \int_{G\setminus U_\delta} \mathrm{d}\lambda(h)|F_\alpha(h)|
\end{aligned}$$

by the choice of δ, Definition 5.1(ii) and Definition 5.2(a). Using condition (iii) in Definition 5.1 we conclude that $\overline{\lim}_\alpha \|F_\alpha \circledast f - f\|_X \leqslant \varepsilon C$. This holds for all $\varepsilon > 0$, hence we obtain (5.10). Finally, (5.11) follows from (5.10) by condition (c) in Definition 5.2. □

Here are some examples of right-homogeneous Banach spaces on G.

(a) X = the space of uniformly continuous, bounded functions on G with the supremum norm.
(b) $X = L^p(G, \mathrm{d}\rho)$, $1 \leqslant p < \infty$, where ρ is a right Haar measure on G.
(c) Let Y be any Banach space of measurable functions on G for which norm convergence implies almost everywhere convergence on G. Then

$$X = \Big\{ f \in Y;\ \text{(i) } R_h f \in Y,\ \forall h \in G,\ \text{(ii) } \|f\|_X := \sup_{h\in G} \|R_h f\|_Y < \infty, \\ \text{(iii) } G \ni h \mapsto R_h f \in X \text{ is continuous} \Big\}$$

is a right-homogeneous Banach space.

Returning to the group $G = \mathrm{AN} \subset \mathrm{Aut}(D),\ D = T(\Omega, \Phi)$ let

$$J = \left\{ \boldsymbol{\nu} = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r;\ \nu_j > \frac{\mathrm{d}}{r} + (j-1)\frac{a}{2}, 1 \leqslant j \leqslant r \right\}$$

be ordered by $\boldsymbol{\nu} \leqslant \boldsymbol{\sigma} \Longleftrightarrow \nu_j \leqslant \sigma_j$ for $1 \leqslant j \leqslant r$. Thus

$$\boldsymbol{\nu} \to \infty \Longleftrightarrow \nu_j \to \infty\ \forall 1 \leqslant j \leqslant r.$$

For all $1 \leqslant j \leqslant r$ and $z, w \in D$ define

$$\delta_j(z, w) = \frac{\Delta_j(\tau(z, z))\Delta_j(\tau(w, w))}{|\Delta_j(\tau(z, w))|^2}.$$

For $\boldsymbol{\nu} \in J$ let

$$\delta_{\boldsymbol{\nu}}(z, w) = \prod_{j=1}^{r} \delta_j(z, w)^{\nu_j - \nu_{j+1}}, \quad z, w \in D,$$

where $\nu_{r+1} := 0$. Thus the Berezin transform $\mathcal{B}^{\boldsymbol{\nu}}$, as an operator of G, is given by

$$(\mathcal{B}^{\nu} f)(g) = \frac{\Gamma_\Omega(\boldsymbol{\nu}^*)}{\Gamma_\Omega(\boldsymbol{\nu} - \frac{\mathrm{d}}{r})} \int_G \mathrm{d}\lambda(h)\delta_{\boldsymbol{\nu}}(g(e), h(e)) f(h),$$

where the left Haar measure $\mathrm{d}\lambda$ is related to the invariant measure on D via

$$\int_G \mathrm{d}\lambda(h) f(h(e)) = \int_D \frac{\mathrm{d}w}{\pi^d} \Delta_{-2d_1/r - d_2/r}(\tau(w, w)) f(w).$$

Define for $\boldsymbol{\nu} \in J$ and $h \in G$

$$F_{\boldsymbol{\nu}}(h) = \frac{\Gamma_\Omega(\boldsymbol{\nu}^*)}{\Gamma_\Omega(\boldsymbol{\nu} - \frac{\mathrm{d}}{r})} \delta_{\boldsymbol{\nu}}(e, h(e)).$$

Our main result in this section is

THEOREM 5.4. $\{F_{\boldsymbol{\nu}}\}_{\boldsymbol{\nu} \in J}$ *is a summability kernel on* $G = \mathrm{AN}$.

By Theorem 5.3, this implies

COROLLARY 5.5 (Correspondence principle for the Berezin transform). *Let* X *be a right-homogeneous Banach space on* $G = \mathrm{AN}$. *Then for every* $f \in X$

$$\lim_{\boldsymbol{\nu} \to \infty} \mathcal{B}^{\boldsymbol{\nu}} f = f$$

in the norm of X *as well as pointwise almost everywhere. Namely,* $\lim_{\boldsymbol{\nu} \to \infty} \mathcal{B}^{\boldsymbol{\nu}} = I$ *in the strong operator topology on* X.

The proof of Theorem 5.4 requires some preparation. Consider for $\boldsymbol{\nu} \in J$ the Hilbert space $H^2_{\boldsymbol{\nu}}(D)$ of holomorphic functions on D (cf. ((4.1)), with reproducing kernel $K^{(\boldsymbol{\nu})}(z,w) = K_w^{(\nu)}(z) = \Delta_{-\boldsymbol{\nu}}(\tau(z,w))$ (cf. (4.2). Thus

$$|K^{(\boldsymbol{\nu})}(z,w)| = \left|\langle K_w^{(\nu)}, K_z^{(\nu)}\rangle_{H^2_{\boldsymbol{\nu}}(D)}\right| \leqslant \|K_w^{(\nu)}\|_{H^2_{\boldsymbol{\nu}}(D)} \cdot \|K_z^{(\nu)}\|_{H^2_{\boldsymbol{\nu}}(D)} \tag{5.12}$$

with equality if and only if $z = w$. This simple fact is used in the following result.

LEMMA 5.6. *For* $1 \leqslant j \leqslant r$

$$\delta_j(z,w) \leqslant 1 \quad \forall z, w \in D.$$

Moreover

$$z = w \Longleftrightarrow \delta_j(z,w) = 1,\ 1 \leqslant j \leqslant r \Longleftrightarrow \delta_r(z,w) = 1.$$

Proof. For every $\boldsymbol{\nu} \in J$ (5.12) yields

$$\delta_{\boldsymbol{\nu}}(z,w) \leqslant 1 \quad \text{and} \quad \delta_\nu(z,w) = 1 \Longleftrightarrow z = w.$$

In particular, if $\nu_1 = \nu_2 = \cdots = \nu_r = \beta > (r-1)\frac{a}{2}$ we obtain

$$\delta_r(z,w)^\beta \leqslant 1 \quad \forall z, w \in D \quad \text{and} \quad \delta_r(z,w)^\beta = 1 \Longleftrightarrow z = w.$$

This implies that

$$\delta_r(z,w) \leqslant 1 \quad \forall z, w \in D \quad \text{and} \quad \delta_r(z,w) = 1 \Longleftrightarrow z = w.$$

Fix $1 \leqslant j \leqslant r-1$. Choose $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_r)$ so that

$$\nu_1 = \nu_2 = \cdots = \nu_j, \qquad \nu_{j+1} = \nu_{j+2} = \cdots = \nu_r > r-1$$
$$\text{and} \quad \nu_j - \nu_{j+1} = \beta\nu_r,$$

where β is a large positive number. Then $\nu_i > (i-1)\frac{a}{2}$ for all i and

$$\delta_{\boldsymbol{\nu}}(z,w) = \prod_{i=1}^r \delta_i^{\nu_i - \nu_{i+1}}(z,w) = \delta_j(z,w)^{\beta\nu_r}\delta_r(z,w)^{\nu_r}.$$

Hence

$$\delta_{\boldsymbol{\nu}}(z,w) = (\delta_j(z,w)^\beta \delta_r(z,w))^{\nu_r} \leqslant 1.$$

Since β is arbitrarily large, we see that

$$\delta_j(z,w) \leqslant 1.$$

For the equality case it suffices to show that if $\delta_r(z,w) = 1$ for $z, w \in D$ then $z = w$. But $\delta_r(z,w) = 1$ implies that for $\beta > (r-1)\frac{a}{2}$ and $\boldsymbol{\nu} = (\beta, \beta, \ldots, \beta)$ we have $\delta_{\boldsymbol{\nu}}(z,w) = 1$. Thus $z = w$ by the reproducing kernel consideration. □

Next, we define a system of open neighborhoods of $\mathbf{1}$ in G by

$$U_j = \{h \in G;\ 1 - \delta < \delta_r(e, h(e))\}, \quad 0 < \delta < 1.$$

Clearly, $\mathbf{1} \in U_\delta$ for all $0 < \delta < 1$ and

$$U_{\delta_1} \subseteq U_{\delta_2} \quad \text{whenever} \quad 0 < \delta_1 < \delta_2 < 1.$$

Also, since $\delta_r(e, h(e)) = \delta_r(\mathbf{1}(e), h(e)) \leqslant 1,\ \forall h \in G$ we have

$$\bigcap_{0<\delta<1} U_\delta = \{h \in G;\ \delta_r(e, h(e)) = 1\} = \{\mathbf{1}\}$$

by Lemma 5.6 and the fact that $G = \text{AN}$ acts on D simply transitively.

Proof of Theorem 5.4. It is clear that $0 \leqslant F_{\boldsymbol{\nu}}(h)$ for all $\boldsymbol{\nu} \in J$ and $h \in G$. Also, by the fact that $\mathcal{B}^{\boldsymbol{\nu}}$ is a stochastic operator, we have

$$\int_G \mathrm{d}\lambda(h) F_{\boldsymbol{\nu}}(h) = 1, \quad \forall \boldsymbol{\nu} \in J.$$

It remains to prove (iii) of Definition 5.1, i.e. that

$$\lim_{\boldsymbol{\nu}\to\infty} \int_{G\setminus U_\delta} \mathrm{d}\lambda(h) F_{\boldsymbol{\nu}}(h) = 0, \quad \forall \delta \in (0, 1).$$

Let $\boldsymbol{\nu} \in J$ and let $\beta = \frac{1}{2}\min\{\nu_j;\ 1 \leqslant j \leqslant r\}$. We assume that $\boldsymbol{\nu}$ is large enough, so that $\boldsymbol{\nu} - \beta \in J$, i.e.

$$\nu_j > \frac{\mathrm{d}}{r} + (j-1)\frac{a}{2} + \frac{1}{2}\min\{\nu_i;\ 1 \leqslant i \leqslant r\}$$

(this is the case, for instance, if $\beta > \frac{\mathrm{d}}{r} + (r-1)\frac{a}{2} = p - 1$). Since

$$\delta_{\boldsymbol{\nu}}(z, w) = \delta_{\boldsymbol{\nu}-\beta}(z, w)\delta_r^\beta(z, w), \quad \forall z, w \in D$$

we obtain

$$\begin{aligned}
&\int_{G\setminus U_\delta} \mathrm{d}\lambda(h) F_{\boldsymbol{\nu}}(h) \\
&\quad = \frac{\Gamma_\Omega(\boldsymbol{\nu}^*)}{\Gamma_\Omega(\boldsymbol{\nu} - \frac{\mathrm{d}}{r})} \int_{\delta_r(e,w)\leqslant 1-\delta} \mathrm{d}\mu_0(w)\delta_{\boldsymbol{\nu}}(e, w) \\
&\quad = \frac{\Gamma_\Omega(\boldsymbol{\nu}^* - \beta)}{\Gamma_\Omega(\boldsymbol{\nu} - \beta - \frac{\mathrm{d}}{r})} \int_{\delta_r(e,w)\leqslant 1-\delta} \mathrm{d}\mu_0(w)\delta_{\boldsymbol{\nu}-\beta}(e, w)\delta_r(e, w)^\beta \times \\
&\qquad \times \frac{\Gamma_\Omega(\boldsymbol{\nu}^*)\Gamma_\Omega(\boldsymbol{\nu} - \beta - \frac{\mathrm{d}}{r})}{\Gamma_\Omega(\boldsymbol{\nu}^* - \beta)\Gamma_\Omega(\boldsymbol{\nu} - \frac{\mathrm{d}}{r})} \\
&\quad \leqslant (1-\delta)^\beta \frac{\Gamma_\Omega(\boldsymbol{\nu}^* - \beta)}{\Gamma_\Omega(\boldsymbol{\nu} - \beta - \frac{\mathrm{d}}{r})} \int_{\delta_r(e,w)\leqslant 1-\delta} \mathrm{d}\mu_0(w)\delta_{\boldsymbol{\nu}-\beta}(e, w) \times
\end{aligned}$$

$$\times \frac{\Gamma_\Omega(\boldsymbol{\nu}^*)\Gamma_\Omega(\boldsymbol{\nu}-\beta-\frac{\mathrm{d}}{r})}{\Gamma_\Omega(\boldsymbol{\nu}^*-\beta)\Gamma_\Omega(\boldsymbol{\nu}-\frac{\mathrm{d}}{r})}$$
$$\leqslant (1-\delta)^\beta \frac{\Gamma_\Omega(\boldsymbol{\nu}^*)\Gamma_\Omega(\boldsymbol{\nu}-\beta-\frac{\mathrm{d}}{r})}{\Gamma_\Omega(\boldsymbol{\nu}^*-\beta)\Gamma_\Omega(\boldsymbol{\nu}-\frac{\mathrm{d}}{r})}.$$

The product formula [3]

$$\Gamma_\Omega(\boldsymbol{\alpha}) = (2\pi)^{\frac{\mathrm{d}_1-r}{2}} \prod_{j=1}^{r} \Gamma\left(\alpha_j - (j-1)\frac{a}{2}\right)$$

together with Stirling's formula imply that

$$\lim_{\boldsymbol{\nu}\to\infty} \frac{\Gamma_\Omega(\boldsymbol{\nu}^*)\Gamma_\Omega(\boldsymbol{\nu}-\beta-\frac{\mathrm{d}}{r})}{\Gamma_\Omega(\boldsymbol{\nu}^*-\beta)\Gamma_\Omega(\boldsymbol{\nu}-\frac{\mathrm{d}}{r})} = 1.$$

Since $\beta = \frac{1}{2}\min\{\nu_j;\ 1 \leqslant j \leqslant r\} \to \infty$ as $\boldsymbol{\nu} \to \infty$, we see that for all $\delta \in (0,1)$

$$\overline{\lim_{\boldsymbol{\nu}\to\infty}} \int_{G\setminus U_\delta} \mathrm{d}\lambda(h) F_{\boldsymbol{\nu}}(h) \leqslant \overline{\lim_{\boldsymbol{\nu}\to\infty}} (1-\delta)^\beta \frac{\Gamma_\Omega(\boldsymbol{\nu}^*)\Gamma_\Omega(\boldsymbol{\nu}-\beta-\frac{\mathrm{d}}{r})}{\Gamma_\Omega(\boldsymbol{\nu}^*-\beta)\Gamma_\Omega(\boldsymbol{\nu}-\frac{\mathrm{d}}{r})} = 0.$$

Thus indeed $\lim_{\boldsymbol{\nu}\to\infty} \int_{G\setminus U_\delta} \mathrm{d}\lambda(h) F_{\boldsymbol{\nu}}(h) = 0$. This completes the proof of Theorem 5.4. □

We will now show that the correspondence principle holds also for the spectral components of the Berezin transform. The spectral component of the Berezin transform $\mathcal{B}^{\boldsymbol{\nu}}$ corresponding to the representation π on $LH^2(\Omega^\# \times V)$ is

$$\mathcal{B}^{\boldsymbol{\nu}}_\pi = \frac{\Gamma_\Omega(\boldsymbol{\nu}^*)}{\Gamma_\Omega(\boldsymbol{\nu}-\frac{\mathrm{d}}{r})} \int_G \mathrm{d}\lambda(g)\delta_{\boldsymbol{\nu}}(e, g(e)) S^{1/2}\pi(g)S^{-1/2}.$$

This leads us to consider the representation

$$\sigma(g) = S^{1/2}\pi(g)S^{-1/2}, \quad g \in G$$

of G on function spaces. Notice that in view of the fact that $(S\psi)(\xi, v) = \Delta^*_{2\rho-d/r}(\xi)\psi(\xi, v)$ we have

$$(\sigma(g)\psi)(\xi, v) = \Delta_{-\rho+d/2r}(\tau(g(e)), g(e))/2)(\tau(g)\psi)(\xi, v)$$

for all $g \in G$ and $(\xi, v) \in \Omega^\# \times V$.

DEFINITION 5.7. A Banach space of measurable functions X on $\Omega^\# \times V$ is *σ-invariant* if

(a) For every $f \in X$ and $g \in G$ we have $\sigma(g)f \in X$ and $\|\sigma(g)f\|_X = \|f\|_X$.
(b) For every $f \in X$ the map $G \ni g \mapsto \sigma(g)f \in X$ is continuous.
(c) Norm convergence in X implies almost everywhere convergence on $\Omega^\# \times V$.

The notion of π-invariant Banach space of measurable functions on $\Omega^{\#} \times V$ is defined analogously. Notice that X is σ-invariant if and only if the space $Y = S^{-1/2}X$ (with the norm $\|F\|_Y = \|S^{1/2}F\|_X$) is π-invariant.

THEOREM 5.8. *Let X be a σ-invariant Banach space of measurable functions on $\Omega^{\#} \times V$. Then for every $f \in X$ the integral defining $\mathcal{B}_\pi^{\boldsymbol{\nu}} f$ converges in the norm of X (as well as almost everywhere on $\Omega^{\#} \times V$) and*

$$\lim_{\boldsymbol{\nu}\to\infty} \|\mathcal{B}_\pi^{\boldsymbol{\nu}} f - f\|_X = 0.$$

The proof is essentially the same as the analogous proof for $\mathcal{B}^{\boldsymbol{\nu}}$, and uses properties (a), (b) and (c) of X and the fact that for all $0 < \delta < 1$

$$\lim_{\boldsymbol{\nu}\to\infty} \frac{\Gamma_\Omega(\boldsymbol{\nu}^*)}{\Gamma_\Omega(\boldsymbol{\nu} - \frac{d}{r})} \int_{G\setminus U_\delta} \mathrm{d}\lambda(g)\delta_{\boldsymbol{\nu}}(e, g(e)) = 0,$$

where $\{U_\delta\}_{0<\delta<1}$ are the neighborhoods of $\mathbf{1}$ in G introduced above.

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