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Invariant symbolic calculi and eigenvalues of invariant operators on symmetric domains

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Abstract

We study the structure of invariant symbolic calculi $\mathcal{A}$ in the context of weighted Bergman spaces on symmetric domains $D = G/K$ and the eigenvalues of the associated link transforms $\mathcal{A}'\mathcal{A}$. We parametrize all such calculi by K-invariants maps which have very simple description. We also introduce and study the properties of the fundamental function $\mathfrak{a}_{\mathcal{A}}(\lambda)$ associated with an invariant symbolic calculus $\mathcal{A}$. Our main result is the formula for the eigenvalues of the associated link transform $\mathcal{A}'\mathcal{A}$:

$$\widetilde{\mathcal{A}'\mathcal{A}}(\underline{\lambda}) = \frac{\mathfrak{a}_{\mathcal{A}}(\underline{\lambda})\,\overline{\mathfrak{a}_{\mathcal{A}}(\overline{\underline{\lambda}})}}{\mathfrak{a}_{\mathcal{T}}(\underline{\lambda})},$$

where $\mathcal{T}$ is the Toeplitz calculus.

0 Introduction

Let $D \equiv G/K$ be a hermitian symmetric domain in $\mathbb{C}^d$ and let $\mathcal{H}$ be a Hilbert space of holomorphic functions on D with reproducing kernel $K(z, w)$, which is invariant under an irreducible projective representation U of G. An *invariant symbolic calculus* $\mathcal{A}$ is a linear map $b \mapsto \mathcal{A}_b$ from a G-invariant subspace $\mathrm{Dom}(\mathcal{A})$ of functions ("symbols") on D into the space $Op(\mathcal{H})$ of operators on $\mathcal{H}$ which intertwines the natural actions of G on symbols and operators:

$$U(g)\mathcal{A}_b\,U(g)^{-1} = \mathcal{A}_{b\circ g^{-1}}, \qquad \forall g \in G, \;\; \forall b \in \mathrm{Dom}(\mathcal{A}).$$

The *adjoint* of $\mathcal{A}$ is the map $\mathcal{A}' : Op(\mathcal{H}) \to \{\text{functions on } D\}$ defined by

$$\langle \mathcal{A}'(T), b\rangle_{L^2(D,\mu_0)} = \langle T, \mathcal{A}_b\rangle_{S_2}, \qquad \forall T \in \mathrm{Dom}(\mathcal{A}'), \quad \forall b \in \mathrm{Dom}(\mathcal{A}),$$

where μ_0 is the G-invariant measure on D and S_2 is the Hilbert-Schmidt class. The operator $\mathcal{B} := \mathcal{A}'\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \to \{\text{functions on } D\}$ is the *link transform* associated with $\mathcal{A}$. It is G-*invariant:* $\mathcal{B}(f \circ g) = (\mathcal{B}f) \circ g$ for all $g \in G$ and $f \in \mathrm{Dom}(\mathcal{B})$, and is therefore diagonalized by the *exponential functions* $\{e_{\underline{\lambda}}\}$ in $\mathrm{Dom}(\mathcal{B})$:

$$\mathcal{B}e_{\underline{\lambda}} = \widetilde{\mathcal{B}}(\underline{\lambda})\, e_{\underline{\lambda}}.$$

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The link transform $\mathcal{B} = \mathcal{A}'\mathcal{A}$ associated with the invariant symbolic calculus $\mathcal{A}$ maps the *active symbol* b of $\mathcal{A}_b$ into its *passive symbol*: $\mathcal{B}(b) = \mathcal{A}'(\mathcal{A}_b)$. For instance, the link transform associated with the *Toeplitz calculus* $\mathcal{T}$ is the well-known *Berezin transform* $\mathcal{B} := \mathcal{T}'\mathcal{T}$, which plays a central role in quantization on symmetric domains.

The link transform and its eigenvalues reflect characteristic features of the underlying quantization procedure. For example, on the cotangent bundle $\mathbb{C}^n = T^*(\mathbb{R}^n)$ the well-known *Weyl calculus* $\mathcal{W}$ is unitary (i.e. $\mathcal{W}'\mathcal{W} = I$), whereas the Toeplitz (or, anti-Wick) calculus $\mathcal{T}$ yields the contraction semi-group of the Laplace operator: $\mathcal{T}'\mathcal{T} = \exp(\beta\Delta)$ for some $\beta > 0$ depending on the underlying Hilbert space. Even more interesting are quantizations of curved symmetric spaces, such as the unit disk and its higher dimensional generalizations (Cartan and Siegel domains).

In this paper we develop a unified approach to compute the eigenvalues $\widetilde{\mathcal{B}}(\underline{\lambda})$ of the link transforms $\mathcal{B}$. This approach is based on a new factorization technique and on a parametrization of the invariant symbolic calculi by K-invariant operators on $\mathcal{H}$ which have very simple structure. Let K_o be the reproducing kernel at the base point $o \equiv K \in G/K$ of D. The formula for the eigenvalues of $\mathcal{B} := \mathcal{A}'\mathcal{A}$ is expressed in terms of the *fundamental function* $\mathfrak{a}_{\mathcal{A}}(\underline{\lambda}) := \langle \mathcal{A}_{e_{\underline{\lambda}}}(K_o), K_o\rangle_{\mathcal{H}}$ of the calculus $\mathcal{A}$ via

$$\widetilde{\mathcal{B}}(\underline{\lambda}) = \frac{\mathfrak{a}_{\mathcal{A}}(\underline{\lambda})\,\overline{\mathfrak{a}_{\mathcal{A}}(\overline{\underline{\lambda}})}}{\mathfrak{a}_{\mathcal{T}}(\underline{\lambda})}.$$

Our approach gives also new proof for the known results concerning the eigenvalues of the Berezin transforms in the context of the weighted Bergman spaces over symmetric domains. For the flat case $D = \mathbb{C}^d$ and the associated weighted Fock spaces $\mathcal{F}_\nu$, we also show that the general approach developed here not only clarifies the relationship between the standard calculi (Toeplitz, Weyl, and Wick calculi) in a very satisfactory manner, but also enables one to construct entirely new invariant symbolic calculi which, nevertheless, can be fully analyzed by closed formulas.

The organization of the paper is as follows. In Section 1 we give first the necessary background on symmetric domains $D = G/K$, Jordan theory, and invariant Hilbert spaces $\mathcal{H}$ of holomorphic functions on D. Subsections 1.2 and 1.3 give the background on the Fock spaces on $\mathbb{C}^d$ and on the weighted Bergman spaces over symmetric tube domains respectively. In particular, we give explicit formulas for the reproducing kernels and the exponential functions in these setups. In section 2 we introduce and study the notions of invariant symbolic calculus ("quantization") and the associated link transform. We establish the one-to-one correspondence between the invariant symbolic calculi and the K-invariant operators on $\mathcal{H}$. The main examples (Toeplitz and Weyl calculi, as well as the calculi associated with the projections onto the K-irreducible subspaces) are discussed. Of special importance is the study of a real-analytic reflection ψ of $D = G/K$ and the induced involution on the Lie algebra $\mathfrak{A}$ of spectral parameters. The main result of this section is Theorem 2.1 which provides a new general formula for the eigenvalues of the link transforms. In Section 3 we introduce the Wick calculus in a general setting via the sesqui-holomorphic extension of real-analytic functions. We also introduce and study the properties of the *fundamental function* associated with an invariant symbolic calculus. Our main result (Theorem 3.1) provides a simple formula for the eigenvalues of the link transform in terms of this fundamental function. The remaining two sections are devoted to applications of this general formula, for weighted Fock spaces over $\mathbb{C}^d$ (Section 4) and weighted Bergman spaces over symmetric tube domains (Section

5). In the flat case we find unexpected relations between the classical functional calculi on $\mathbb{C}^d$, and present a new class of calculi which are still explicitly solvable. For symmetric tube domains, we present a new proof for the spectral analysis of the Berezin transform.

It should be remarked that our formulas for the eigenvalues of link transforms can be translated into corresponding formulas describing the link transforms in terms of canonical sets of generators of the ring of invariant differential operators on D (via the Harish-Chandra transform).

The general theory of invariant symbolic calculi on symmetric domains developed here may be applied in a much wider setting. For example - in the context of real symmetric domains our approach led to some definite results, particularly in the rank-one case (see [AU01]). It is possible and very interesting to develop our theory in the context of the invariant Hilbert spaces of holomorphic functions on symmetric Siegel domains, which are associated with the Wallach parameter but are not part of the holomorphic discrete series. The discrete Wallach points are of special interest (and most difficult to handle), since they are the only points which survive when the rank of the domains becomes infinite. The integral formulas for the invariant inner products associated with the Wallach set obtained in [AU97], [AU98] and [AU99] will be very useful in this goal. Finally, our theory can also be developed in the context of NA-invariant and vector-valued Hilbert spaces of holomorphic functions on symmetric domains.

1 Invariant Hilbert spaces of holomorphic functions on symmetric domains

1.1 The general framework

Let D be an irreducible hermitian symmetric domain in $\mathbb{C}^d$ with a distinguished base point $o \in D$ (called the "origin"). Then D can be realized as the quotient $D = G/K$, where $G := \mathrm{Iso}(D) \subseteq \mathrm{Aut}(D)$ is the real Lie group of all biholomorphic Riemannian isometries of D, and $K := \{g \in G; g(o) = o\}$ is the maximal compact subgroup of G. Our main applications are in the cases where D is either $\mathbb{C}^d$, or irreducible symmetric tube domain (i.e. a symmetric Siegel domain of type I), see subsections 1.2 and 1.3 bellow. Nevertheless, we prefer to develop our theory in the general setup, so as to allow future applications (for instance, to general symmetric hermitian spaces). [He78] and [He84] are our general references for symmetric spaces and semi-simple Lie groups. [Hu63], [Gi64] and [FK94] are the general references for analysis on symmetric domains, and [Lo77], [Up87] are the references to the analysis on symmetric domains from the Jordan-theoretic point of view. [UU96] and [Un98] are general references for pseudo-differential analysis on symmetric cones and quantization respectively.

It is known that for every $z \in D$ there is a unique *symmetry* $s_z \in G$ (i.e. $s_z \circ s_z = 1_G$, the unit of G) for which z is the unique fixed point. Moreover,

$$s_z = g \circ s_o \circ g^{-1}, \quad \forall g \in G,\ g(o) = z. \tag{1.1}$$

The *Cartan involution* induced by s_o, $\Theta(g) := s_o \circ g \circ s_o$, $g \in G$ gives rise to the *Iwasawa* *decomposition*

$$G = NAK \tag{1.2}$$

in which A and N are maximal abelian and maximal nilpotent subgroups of G respectively. NA is a maximal solvable subgroup of G, and the evaluation map $NA \ni g \mapsto g(o) \in D$ is a surjective Riemannian isometry. Thus, for every $z \in D$ there exists a unique element $g_z \in NA$ for which $g_z(o) = z$. In what follows we shall use the important map $\psi : D \to D$ which is defined by

$$\psi(z) := g_z^{-1}(o). \tag{1.3}$$

We shall show that ψ is a real-analytic diffeomorphism of period 2 of D whose unique fixed point is the origin o. It is known that any pair of points in D can be joined by a unique geodesic line. Also, since the elements of G act on D as Riemannian isometries, they permute the geodesic lines in D. In particular, the symmetry s_z maps each geodesic line through z into itself, reverses its orientation and preserves distances. Given $w \in D$, let z be the mid-point along the geodesic line between o and z. We denote $\varphi_w := s_z$. Thus

$$\varphi_w \circ \varphi_w = 1_G, \quad \varphi_w(o) = w, \quad \text{and} \quad \varphi_w(w) = o.$$

Let $\mathfrak{A}$ be the Lie algebra of A. It is isomorphic to $\mathbb{R}^r$, where r is the *rank* of D. Given $g \in G$ with Iwasawa decomposition $g = n\,a\,k$ (with $n \in N, a \in A, k \in K$) let $\mathfrak{A}(g) \in \mathfrak{A}$ be the unique element for which $\exp \mathfrak{A}(g) = a$. Then $\mathfrak{A}(n_1 g k_1) = \mathfrak{A}(g)$ $\forall n_1 \in N$, $\forall k_1 \in K$, and $\mathfrak{A}(a_1 a_2) = \mathfrak{A}(a_1) + \mathfrak{A}(a_2)$ $\forall a_1, a_2 \in A$. Let $\mathfrak{A}^{*\mathbb{C}} \equiv \mathbb{C}^r$ be the complexification of the dual of $\mathfrak{A}$ and let $\underline{\rho} \in \mathfrak{A}^*$ be the half sum of the positive roots. The *exponential functions*

$$e_{\underline{\lambda}}(z) := \exp\langle \mathfrak{A}(g) | \underline{\lambda} + \underline{\rho} \rangle, \qquad g \in G, \;\; g(o) = z, \tag{1.4}$$

are N-invariant functions on D which are the eigenfunctions of NA

$$e_{\underline{\lambda}}(h(z)) = e_{\underline{\lambda}}(h(o))\, e_{\underline{\lambda}}(z), \qquad \forall h \in NA, \;\; \forall z \in D. \tag{1.5}$$

Let us define

$$\mathcal{X}_{\underline{\lambda}} = \overline{\operatorname{span}}\{e_{\underline{\lambda}} \circ g; g \in G\} \tag{1.6}$$

where the closure is taken in the topology of uniform convergence on compact subsets of D. It is known that $\operatorname{span}\{\mathcal{X}_{\underline{\lambda}}; \underline{\lambda} \in \mathbb{C}^r\}$ is dense in $C^\infty(D)$ in that topology. The *spherical function* associated with the exponential function $e_{\underline{\lambda}}$ is

$$\phi_{\underline{\lambda}}(z) := \int_K e_{\underline{\lambda}}(k(z))\, dk.$$

Then $\phi_{\underline{\lambda}} \in \mathcal{X}_{\underline{\lambda}}$ is K-invariant, $\phi_{\underline{\lambda}}(o) = e_{\underline{\lambda}}(o) = 1$, and for every $f \in \mathcal{X}_{\underline{\lambda}}$

$$\int_K f(k(z))\, dk = f(o)\, \phi_{\underline{\lambda}}(z), \quad \forall z \in D. \tag{1.7}$$

Let W be the *Weyl group* of D. It is a subgroup of $GL(\mathbb{C}^r)$ which contains the permutation group, and moreover

$$\phi_{\underline{\lambda}} = \phi_{\underline{\lambda}'} \quad \text{if and only if there exists} \quad w \in W \quad \text{so that} \quad \underline{\lambda} = w(\underline{\lambda}'). \tag{1.8}$$

It follows that

$$\mathcal{X}_{\underline{\lambda}} = \mathcal{X}_{\underline{\lambda}'} \quad \text{if and only if there exists} \quad w \in W \quad \text{so that} \quad \underline{\lambda} = w(\underline{\lambda}').$$

We denote by $\mathrm{Diff}(D)^G$ the algebra of *G-invariant differential operators* on D (i.e. differential operators T on D so that $T(f \circ g) = T(f) \circ g$ for all $g \in G$ and all $f \in C^\infty(D)$). A fundamental property of the exponential and spherical functions is that they are *joint eigen-functions for* $\mathrm{Diff}(D)^G$:

$$T(\phi_{\underline{\lambda}}) = \widetilde{T}(\underline{\lambda})\,\phi_{\underline{\lambda}}, \qquad T(e_{\underline{\lambda}}) = \widetilde{T}(\underline{\lambda})\,e_{\underline{\lambda}}, \qquad \forall T \in \mathrm{Diff}(D)^G,\ \forall \underline{\lambda} \in \mathbb{C}^r. \tag{1.9}$$

A fundamental result of Harish-Chandra is that the eigen-value map $T \mapsto \widetilde{T}(\underline{\lambda})$ (called the *Harish-Chandra transform*) is an algebra isomorphism of $\mathrm{Diff}(D)^G$ onto the algebra $\mathbb{C}[x_1, \ldots, x_r]^W$ of W-invariant polynomials in $x_1, \ldots, x_r$. In particular, $\mathrm{Diff}(D)^G$ is commutative.

Using standard tools from spectral theory, this result extends to more general (bounded or unbounded) G-invariant operators on D (see [E98]). Thus, if T is a G-invariant operator, defined on a G-invariant subspace $\mathcal{X}$ of functions on D, then (1.9) holds whenever $e_{\underline{\lambda}} \in \mathcal{X}$, and the generalized Harish-Chandra transform $T \mapsto \widetilde{T}(\cdot)$ is injective. Thus T is uniquely determined by its eigenvalues $\widetilde{T}(\underline{\lambda})$, and they can be computed by

$$\widetilde{T}(\underline{\lambda}) = T(e_{\underline{\lambda}})(o) = T(\phi_{\underline{\lambda}})(o). \tag{1.10}$$

In what follows we shall use (1.10) with the exponential function, since they are much simpler than the spherical functions, and obey (1.5).

Let μ_0 be the (unique up to a multiplicative constant) *G-invariant measure* on D, i.e.

$$\int_D f\, d\mu_0 = \int_G f(g(o))\, dg.$$

The normalization of μ_0 will be fixed in the definition (1.21) below. For any Borel measure μ on D we define a function $\tilde{\mu}$ by

$$\mathrm{Dom}(\tilde{\mu}) := \{\underline{\lambda} \in \mathbb{C}^r ; e_{\underline{\lambda}} \in L^1(D, \mu)\}$$

and

$$\tilde{\mu}(\underline{\lambda}) := \int_D e_{\underline{\lambda}}(z)\, d\mu(z), \quad \forall \underline{\lambda} \in \mathrm{Dom}(\tilde{\mu}).$$

Notice that if μ is K-invariant then $\tilde{\mu}$ is its (spherical) *Fourier transform* and

$$\tilde{\mu}(\underline{\lambda}) := \int_D \phi_{\underline{\lambda}}(z)\, d\mu(z), \quad \forall \underline{\lambda} \in \mathrm{Dom}(\tilde{\mu}).$$

In particular, if f is a measurable function on D then $\tilde{f} := \widetilde{f\, d\mu_0}$, i.e.

$$\tilde{f}(\underline{\lambda}) := \int_D e_{\underline{\lambda}}(z)\, f(z)\, d\mu_0(z). \tag{1.11}$$

Again, if f is K-invariant then $\tilde{f}$ is its (spherical) Fourier transform,

$$\tilde{f}(\underline{\lambda}) := \int_D \phi_{\underline{\lambda}}(z)\, f(z)\, d\mu_0(z) \quad \forall \underline{\lambda} \in \mathrm{Dom}(\tilde{f}).$$

Next, if μ is a K-invariant measure on D then the *convolution operator*

$$(C_\mu f)(z) := \int_D f(g(w))\, d\mu(w), \quad \text{where } g \in G \text{ and } g(o) = z,$$

is well-defined (on an appropriate domain) and G-invariant. Its Harish-Chandra transform is the Fourier transform of μ:

$$\widetilde{C_\mu}(\underline{\lambda}) = C_\mu(e_{\underline{\lambda}})(o) = \int_D e_{\underline{\lambda}}\, d\mu = \tilde{\mu}(\underline{\lambda}).$$

Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on D on which the point evaluation functionals $\mathcal{H} \ni f \mapsto f(w)$, $w \in D$, are continuous, and let $K(z,w) = K_w(z)$ be the *reproducing kernel* of $\mathcal{H}$. Let

$$k_w(z) := \frac{K_w(z)}{\|K_w\|} = \frac{K(z,w)}{K(w,w)^{1/2}}$$

be the normalized kernel at the point $w \in D$. For convenience we adopt the normalization

$$K(o,o) = 1.$$

We assume that G acts on $\mathcal{H}$ by means of an irreducible *projective representation* U of the form:

$$U(g)(f)(z) := j(g^{-1}, z)\, f(g^{-1}(z)), \quad \forall g \in G, \ \forall z \in D. \tag{1.12}$$

Thus, each operator $U(g)$ is isometric on $\mathcal{H}$, the function $j(g^{-1}, z)$ is holomorphic in z for all $g \in G$, and

$$j(g_2^{-1} \circ g_1^{-1}, z) = c(g_1, g_2)\, j(g_2^{-1}, g_1^{-1}(z))\, j(g_1^{-1}, z), \quad \forall g_1, g_2 \in G, \ \forall z \in D, \tag{1.13}$$

where $c(g_1, g_2)$ is a unimodular constant depending only on g_1, g_2. This is clearly equivalent to

$$U(g_1 \circ g_2) = c(g_1, g_2)\, U(g_1)\, U(g_2), \quad \forall g_1, g_2 \in G. \tag{1.14}$$

The relationship (1.13) yields also

$$|j(g_2^{-1} \circ g_1^{-1}, z)| = |j(g_2^{-1}, g_1^{-1}(z))|\, |j(g_1^{-1}, z)|, \quad \forall g_1, g_2 \in G, \ \forall z \in D.$$

In particular,

$$|j(k_1 \circ k_2, o)| = |j(k_1, o)|\, |j(k_2, o)|, \quad \forall k_1, k_2 \in K.$$

This fact and the compactness of K yield

$$|j(k, o)| = 1, \quad \forall k \in K. \tag{1.15}$$

Also, (1.13) and the fact that $j(1_G, z) \equiv 1$ lead to $c(g, g^{-1}) = 1$, i.e.

$$j(g^{-1}, g(z))\, j(g, z) = 1, \quad \forall g \in G, \ z \in D. \tag{1.16}$$

This implies that $U(g^{-1}) = U(g)^{-1}$ for all $g \in G$, and that $c(g_1, g_2)\, c(g_2^{-1}, g_1^{-1}) = 1$ for all $g_1, g_2 \in G$. Thus the natural action of G on operators on $\mathcal{H}$,

$$\pi(g)(T) := U(g)\, T\, U(g)^{-1}, \quad g \in G,$$

is a genuine representation, i.e. $\pi(g_1 \circ g_2) = \pi(g_1)\,\pi(g_2), \quad \forall g_1, g_2 \in G$.

The fact that the operators $U(g)$, $g \in G$, are unitary on $\mathcal{H}$ is reflected in the transformation formula of the reproducing kernel:

$$j(g,z)\; K(g(z), g(w))\; \overline{j(g,w)} = K(z,w), \quad \forall g \in G, \;\; \forall z, w \in D, \tag{1.17}$$

a fact which can also be written as $(U(g) \otimes \overline{U(g)})(K) = K$. Also, (1.17) can be written as

$$U(g)(K_w) = \overline{j(g,w)}\, K_{g(w)}, \quad \forall g \in G, \; \forall w \in D. \tag{1.18}$$

Notice that since $K(o,o) = 1$, (1.17) implies in particular that

$$|j(g,o)|^{-2} = K(z,z), \quad \text{where } g \in G, \;\; g(o) = z. \tag{1.19}$$

Of particular interest is the case where $\mathcal{H}$ is the *weighted Bergman space*

$$\mathcal{H} := L^2_a(D,\mu) = L^2(D,\mu) \cap \{\text{holomorphic functions}\}$$

with respect to a K-invariant absolutely continuous measure μ on D. In this case we require that the measure μ transforms under G via

$$d\mu(g(z)) = |j(g,z)|^2\, d\mu(z), \quad \forall g \in G. \tag{1.20}$$

Thus U extends to a projective representation of $L^2(D,\mu)$. The transformation rule (1.20) of the measure μ and (1.15) yield easily that the measure μ_0 defined by

$$d\mu_0(z) := |j(g,o)|^{-2}\, d\mu(z) = K(z,z)\, d\mu(z), \quad \text{where } \; g \in G \; \text{ and } \; g(o) = z \tag{1.21}$$

is well-defined (independent of the particular $g \in G$ satisfying $g(o) = z$), and is *G-invariant*, i.e.

$$d\mu_0(g(z)) = d\mu_0(z), \quad \forall g \in G.$$

1.2 Weighted Fock spaces on $\mathbb{C}^d$

For $\nu > 0$ consider on $\mathbb{C}^d$ the probability measure

$$d\mu_\nu(z) = \left(\frac{\nu}{\pi}\right)^d e^{-\nu|z|^2}\, dm(z),$$

where $dm(z)$ is Lebesgue measure. The *weighted Fock space*

$$\mathcal{F}_\nu = \mathcal{F}_\nu(\mathbb{C}^d) = L^2_a(\mathbb{C}^d, \mu_\nu) := L^2(\mathbb{C}^d, \mu_\nu) \cap \{\text{holomorphic functions}\}$$

has a reproducing kernel

$$K^{(\nu)}(z,w) = e^{\nu\langle z,w\rangle}.$$

Let G be the semi-direct product of the unitary group $U(d)$ and the translation group $T := \{g_w;\; w \in \mathbb{C}^d\} \equiv \mathbb{C}^d$, where $g_w(z) = z + w$. Notice that in the Iwasawa decomposition of G we have $K = U(d)$, $A = \{1_G\}$, $N = T$, and that $U(d)$ normalizes T:

$$k\, g_w\, k^{-1} = g_{k(w)}, \quad \forall\, k \in U(d), \; \forall\, w \in \mathbb{C}^d.$$

G acts isometrically on $L^2(\mathbb{C}^d, \mu_\nu)$ by the rule

$$U^{(\nu)}(g)\ f(z) = j(g^{-1}, z)\, f(g^{-1}(z)),$$

where

$$\begin{aligned} j(g^{-1}, z) &= \exp\{-\frac{\nu}{2}\,|g(0)|^2 + \nu\langle z, g(0)\rangle\} \\ &= K^{(\nu)}(z, g(0))/K^{(\nu)}(g(0), g(0)) = k^{(\nu)}_{g(0)}(z). \end{aligned}$$

Here $k^{(\nu)}_w(z) := K^{(\nu)}(z, w)/K^{(\nu)}(w, w)^{1/2}$. Indeed, for $g \in G$

$$d\mu_\nu(g(z)) = |j(g, z)|^2\, d\mu_\nu(z), \tag{1.22}$$

i.e. (1.20) holds. To prove (1.22) let us write $g = g_w\, k$, where $k \in U(d)$ and $g_w \in T$. Then

$$\begin{aligned} d\mu_\nu(g(z)) &= (\nu/\pi)^d\ e^{-\nu\,|k\,z+w|^2}\ dm(k\,z + w) \\ &= (\nu/\pi)^d\ e^{-\nu\,|z|^2}\ e^{-2\nu\ Re\,\langle z, k^*(w)\rangle - \nu\,|w|^2}\ dm(z) \\ &= \left|e^{-\nu\,\langle z, k^*(w)\rangle\ -\frac{1}{2}\ \nu|w|^2}\right|^2\ d\mu_\nu(z) \\ &= \left|e^{\nu\,\langle z, g^{-1}(0)\rangle\ -\frac{\nu}{2}\ |g^{-1}(0)|^2}\right|^2\ d\mu_\nu(z) = |j(g, z)|^2\, d\mu_\nu(z). \end{aligned}$$

The action $U^{(\nu)}$ of G is an irreducible *projective representation* (see (1.14)) since, by elementary calculations, one obtains (1.13) with the unimodular constant

$$c(g_1, g_2) := \exp\left\{i\,\nu\,\mathrm{Im}\langle g_1(0), g_1'(0)(g_2(0))\rangle\right\}.$$

Let us define

$$d\mu_0(w) = \left(\frac{\nu}{\pi}\right)^d\ dm(w).$$

Then μ_0 is clearly G-invariant. The reason for the particular normalization is the following relationship between μ_0 and μ_ν

$$d\mu_0(z) = |j(g, 0)|^{-2}\, d\mu_\nu(z), \qquad g \in G, g(0) = z,$$

in accordance with (1.21). The ring of G-invariant differential operators is simply the polynomial ring $\mathbb{C}[\Delta]$, where

$$\Delta = \langle\partial|\partial\rangle = \sum_{j=1}^{d} \partial_j\ \overline{\partial}_j$$

is the Euclidean Laplacian (properly normalized). The exponential functions

$$e_{a,b}(z) = e^{\langle z|b\rangle}\ e^{\langle a|z\rangle}, \quad a, b \in \mathbb{C}^d \tag{1.23}$$

are eigen-functions of Δ:

$$\Delta(e_{a,b}) = \lambda\ e_{a,b} \quad \text{with} \quad \lambda = \langle a|b\rangle$$

as well as of the translation subgroup T:

$$e_{a,b} \circ g_w = e_{a,b}(w)\, e_{a,b}, \qquad \forall w \in \mathbb{C}^d.$$

Notice that $e_{a,b}$ is bounded if and only if $b = -a$, and in this case $e_{a,-a}(z) = e^{2i\,\mathrm{Im}\,\langle a|z\rangle}$ and $\lambda = \langle a, -a\rangle = -\|a\|^2$. The map $\psi(w) = g_w^{-1}(0)$ is simply $\psi(w) = -w = s_0(w)$ and the exponential functions transform under ψ according to the rule

$$e_{a,b} \circ \psi = e_{-a,-b}. \tag{1.24}$$

Since $\overline{e_{a,b}} = e_{b,a}$, it follows that

$$\overline{e_{a,b} \circ \psi} = e_{-b,-a}. \tag{1.25}$$

In this case the spherical function ϕ_λ associated with $e_{a,b}$ $(\lambda = \langle a|b\rangle)$ is calculated explicitly

$$\phi_\lambda(z) = \int\limits_{U(d)} e_{a,b}(k(z))\; dk = \sum_{\ell=0}^{\infty} \frac{1}{(d)_\ell} \frac{\lambda^\ell \|z\|^{2\ell}}{\ell!} = {}_0F_1(d;\; \lambda\,\|z\|^2).$$

1.3 Weighted Bergman spaces over symmetric tube domains

Let X be an irreducible Euclidean Jordan algebra of dimension d with unit element e, and let $\Omega = \{x^2;\; X \ni x \text{ invertible}\}$ be the associated symmetric cone. The *triple product* $\{x, y, z\} = (xy)z + (yz)x - (zx)y$ extends to the complexification $Z = X^{\mathbb{C}} = X \oplus iX$, and Z carries the structure of a *JB^*-algebra* with product $zw = \{z, e, w\}$, unit e, and involution $z^* = \{e, z, e\}$ (see [Up87]). The *tube domain* associated with Ω is

$$T(\Omega) := X + i\Omega = \{z \in Z;\; \frac{z - z^*}{2i} \in \Omega\}\;.$$

It is well known that $T(\Omega)$ is an irreducible hermitian symmetric domain (symmetric Siegel domain of type I). Thus with respect to the Bergman metric the holomorphic symmetry at ie, $s_{ie}(z) = -z^{-1}$, is an isometry, and the group $G := \mathrm{Aut}\,(T(\Omega))$ of all biholomorphic automorphisms of $T(\Omega)$ acts on it transitively. Let $G = NAK$ be the *Iwasawa decomposition* with respect to the Cartan involution $g \mapsto s_{ie}\; g\; s_{ie}$. Thus $K = \{g \in G;\; g(ie) = ie\}$ is a maximal compact subgroup of G and $T(\Omega) \equiv G/K \equiv NA$, with A maximal abelian and N maximal nilpotent subgroups of G. Since NA acts on $T(\Omega)$ simply transitively, for any $z \in T(\Omega)$ there exists a unique element $g_z \in NA$ so that $g_z(ie) = z$.

Let $N(z)$ and $tr(z) = \langle z|e\rangle$ be the *determinant* ("*norm*") and *trace* polynomials (defined on X via the spectral theorem, and extended linearly to the complexification $Z = X^{\mathbb{C}}$). Fix a frame $\{e_j\}_{j=1}^r$ of pairwise orthogonal primitive idempotents in X, where r is the *rank* of X (also, the rank of $T(\Omega)$). Thus $e = \sum\limits_{j=1}^{r} e_j$, and Z has a *Peirce decomposition*

$$Z = \sum_{1 \le i \le j \le r}^{\oplus} Z_{i,j}$$

relative to $\{e_j\}_{j=1}^r$ (see [Lo77], [FK94], and [Up87]). For $1 \le k \le r$ let N_k be the determinant polynomial of the JB^*-sub-algebra

$$Z_k := \sum_{1 \le i \le j \le k}^{\oplus} Z_{i,j}$$

whose unit is $u_k = \sum_{j=1}^{k} e_j$. It is known that the *characteristic multiplicity*

$$a = \dim_{\mathbb{C}}\ Z_{i,j}, \quad 1 \leq i < j \leq r$$

is independent of the frame and of the pair (i,j) with $i<j$. Let P be the orthogonal projection from Z onto Z_k and extend N_k to all of Z via $N_k(z) = N_k(P_k(z))$. Clearly, $N_r = N$.

The *conical function* associated with $\boldsymbol{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ is defined on $T(\Omega)$ via

$$N_{\boldsymbol{s}}(z) := N_1(z)^{s_1 - s_2} N_2(z)^{s_2 - s_3} \cdots N_r(z)^{s_r}, \qquad z \in T(\Omega).$$

Notice that if $\boldsymbol{s} \in \mathbb{N}^r$ and $\boldsymbol{s} \geq 0$ in the sense that $s_1 \geq s_2 \geq \cdots \geq s_r \geq 0$ then $N_{\boldsymbol{s}}(z)$ is a polynomial.

The *Gindikin-Koecher Gamma function* is defined for $\boldsymbol{s} \in \mathbb{C}^r$ with $Re\,(s_j) > (j-1)\,\frac{a}{2}\ \forall j$ by the absolutely convergent integral

$$\Gamma_\Omega(\boldsymbol{s}) = \int_\Omega e^{-tr(x)} N_{\boldsymbol{s}}(x)\, d\mu_\Omega(x)$$

where $d\mu_\Omega(x) = N(x)^{-\frac{d}{r}} dx$ is the (unique up to a constant multiple) measure on Ω which is invariant under $GL(\Omega) := \{g \in GL(X); G(\Omega) = \Omega\}$. It is known (see [Gi64] and [FK94]) that

$$\Gamma_\Omega(\boldsymbol{s}) = (2\pi)^{\frac{d-r}{2}} \prod_{j=1}^{r} \Gamma\left(s_j - (j-1)\,\frac{a}{2}\right) . \tag{1.26}$$

This allows to extend $\Gamma_\Omega(\boldsymbol{s})$ to an entire meromorphic function on $\mathbb{C}^r$. Let us denote

$$\tau(z,w) := \frac{z - w^*}{2i}, \quad z, w \in Z \quad \text{and} \quad \tau(z) := \tau(z,z) = \operatorname{Im}(z).$$

The functions $N_{\boldsymbol{s}} \circ \tau$ are *joint eigenfunctions* of the group NA:

$$N_{\boldsymbol{s}}(\tau(g(z))) = N_{\boldsymbol{s}}(\tau(g(ie)))\, N_{\boldsymbol{s}}(\tau(z)), \quad \forall\, g \in NA. \tag{1.27}$$

See [UU94]. Thus the exponential functions in this context are given by

$$e_{\underline{\lambda}}(z) := N_{\underline{\lambda}+\underline{\rho}}(\tau(z,z)), \quad z \in T(\Omega), \tag{1.28}$$

where $\underline{\rho} := (\frac{1}{2}((j-1)a+1))_{j=1}^{r}$ is the half-sum of the positive roots. Property (1.27) allows the derivation of the fundamental formula

$$\frac{1}{\Gamma_\Omega(\boldsymbol{s})} \int_\Omega e^{-\langle \frac{z}{i} | x \rangle} N_{\boldsymbol{s}}(x) d\mu_\Omega(x) = N_{\boldsymbol{s}}\left(\left(\frac{z}{i}\right)^{-1}\right) = N^*_{-\boldsymbol{s}^*}(\tfrac{z}{i}) \tag{1.29}$$

valid for all $\boldsymbol{s} \in \mathbb{C}^r$ with $Re\,(s_j) > (j-1)\,\frac{a}{2}\quad \forall j$, and all $z \in T(\Omega)$. Here, $\boldsymbol{s}^* = (s_r, s_{r-1}, s_{r-2}, \cdots, s_1)$ and $N^*_{\boldsymbol{\alpha}}$ are the conical functions associated with the frame in reverse order $\{e_r, e_{r-1}, e_{r-2}, \cdots, e_1\}$.

The *Wallach set* $W(D)$ is the set of all $\nu \in \mathbb{C}$ for which the function

$$K^{(\nu)}(z,w) := N(\tau(z,w))^{-\nu}, \quad z, w \in T(\Omega) \tag{1.30}$$

is *positive definite*. It is known (see [Be75], [Gi75],[VR76], [W79], [La86], [La87], [FK90]) that

$$W(D) = \{0, \frac{a}{2}, 2\frac{a}{2}, \ldots, (r-1)\frac{a}{2}\} \cup ((r-1)\frac{a}{2}, \infty).$$

For $\nu \in W(D)$ let $\mathcal{H}_\nu$ be the Hilbert space of analytic functions on $T(\Omega)$ whose reproducing kernel is $K^{(\nu)}(z,w)$. Let $p = 2\frac{d}{r} = (r-1)\,a + 2$ be the *genus* of $T(\Omega)$. Then (with J denoting the Jacobian)

$$(J\,g(z))^{\nu/p}\; K^{(\nu)}(g(z), g(w))\; \overline{Jg(w)}^{\nu/p} = K^{(\nu)}(z,w), \quad \forall\, g \in G, \quad \forall\, z, w \in T(\Omega)\ .$$

Thus G acts isometrically on $\mathcal{H}_\nu$ via

$$U^{(\nu)}(g)\, f = (J(g^{-1}))^{\nu/p} \cdot (f \circ g^{-1}), \qquad g \in G,\ f \in \mathcal{H}_\nu,$$

and this action is an irreducible *projective representation* of G which becomes a unitary representation when $\nu \in W(T(\Omega)) \cap \frac{1}{2}\mathbb{N}$. Thus, in the notation of subsection 1.1,

$$j(g,z) := (Jg(z))^{\frac{\nu}{p}}, \quad \forall g \in G, \ \forall z \in T(\Omega).$$

If $\nu > p-1$ then $\mathcal{H}_\nu$ is the *weighted Bergman space*

$$\mathcal{H}_\nu = L^2_a(T(\Omega), \mu_\nu) = L^2(T(\Omega), \mu_\nu) \cap \ \{\text{holomorphic functions}\}$$

where

$$d\mu_\nu(z) := a(\nu)\ N(\tau(z))^{\nu-p} dm(z),$$

$dm(z)$ being the Lebesgue measure, and

$$a(\nu) := \frac{\Gamma_\Omega(\nu)}{(4\pi)^d\,\Gamma_\Omega(\nu - \frac{d}{r})}.$$

The measure μ_ν transforms according to the rule

$$d\mu_\nu(g(z)) = \left|Jg(z)\right|^{\frac{2\nu}{p}} d\mu_\nu(z), \tag{1.31}$$

in accordance with (1.20). In particular, the measure $N(\tau(z))^{-p}\ dm(z)$ is the unique, up to a multiplicative factor, *G-invariant measure* on $T(\Omega)$. It will be convenient for us to use the following normalization for the G-invariant measure:

$$d\mu_0(z) := a(\nu)\, N(\tau(z))^{-p}\ dm(z).$$

With this normalization, the measures μ_0 and μ_ν are related by

$$d\mu_0(z) = |Jg(ie)|^{-\frac{2\nu}{p}}\ d\mu_\nu(z), \quad g \in G,\ g(ie) = z,$$

in accordance with (1.21).

2.1 Invariant symbolic calculi and covariant fields of operators

As in section 1, let D be a hermitian symmetric domain in $\mathbb{C}^d$ and let $\mathcal{H}$ be a Hilbert space of holomorphic functions on D with reproducing kernel $K(z,w) = K_w(z)$, on which the group $G := \mathrm{Iso}(D)$ of all biholomorphic Riemannian isometries of D acts isometrically and irreducibly by means of a projective representation U with multiplier $j(g,z)$ (1.12).

Definition 2.1 *An* invariant symbolic calculus *is a map $b \mapsto \mathcal{A}_b$ from a G-invariant subspace $Dom(\mathcal{A})$ of functions on D into the space $Op(\mathcal{H})$ of closed operators on $\mathcal{H}$, such that*

1. *$span\{\mathcal{X}_{\underline{\lambda}} \cap Dom(\mathcal{A}); \underline{\lambda} \in \mathbb{C}^r\} = span\{e_{\underline{\lambda}} \circ g; g \in G, \underline{\lambda} \in \mathbb{C}^r\}$ is dense in $Dom(\mathcal{A})$ in the topology of uniform convergence on compact subsets.*

2. *$span\{K_w; w \in D\} \subset Dom(\mathcal{A}_b)$, $\forall b \in Dom(\mathcal{A})$;*

3. *$\mathcal{A}$ intertwines the natural group actions on functions and operators:*

$$U(g)\mathcal{A}_b U(g)^{-1} = \mathcal{A}_{b\circ g^{-1}}, \quad \forall\, g \in G, \forall\, b \in Dom\,(\mathcal{A}). \tag{2.32}$$

The function b is called the *active symbol* (or, strong symbol) of the operator $\mathcal{A}_b$. The invariant symbolic calculi appear naturally in Berezin's theory of *quantization* on symmetric domains, see [Be71], [Be72], [Be73], [Be74-1], [Be74-2], [Be75] and [Be78].

Remarks: (i) If the function 1 belongs to $\mathrm{Dom}(\mathcal{A})$ then $\mathcal{A}_1$ commutes with all the operators $U(g)$, $g \in G$. Since U is irreducible, $\mathcal{A}_1$ is a multiple of the identity operator. In this case we normalize $\mathcal{A}$ by requiring that

$$\mathcal{A}_1 = I.$$

(ii) We are vague here about the nature of the functions (symbols) in $\mathrm{Dom}(\mathcal{A})$ and the nature of the operators $\mathcal{A}_b$. In the applications the symbols are measurable (or even continuous) functions, and the operators are bounded.
(iii) One can study the more general setup in which $\mathcal{A}_b$ is an operator from $\mathcal{H}$ into another Hilbert space $\mathcal{L}$ of functions (holomorphic or not) on D which is invariant under an isometric action V of G. The intertwining property is then

$$V(g)\mathcal{A}_b U(g)^{-1} = \mathcal{A}_{b\circ g^{-1}} \qquad \forall\, g \in G, \forall\, b \in \mathrm{Dom}\,(\mathcal{A}).$$

The space $\mathcal{L}$ need not be irreducible. For instance, let D be an irreducible hermitian symmetric domain (realized either as a Cartan or as a Siegel domain), let $\nu > p-1$ and let $\mathcal{L} := L^2(D,\mu_\nu)$ and $\mathcal{H} := L^2_a(D,\mu_\nu)$, the space of holomorphic functions in $L^2(D,\mu_\nu)$. $G = \mathrm{Aut}(D)$ acts isometrically on both $\mathcal{L}$ and $\mathcal{H}$ via $U^{(\nu)}(g)(f) := (J(g^{-1})(z))^{\nu/p} f \circ g^{-1}$. The study of *Hankel operators* and their generalizations [A96] fits in naturally here.

Another important case is when $\mathcal{H}$ and $\mathcal{L}$ are both Hilbert spaces of holomorphic functions on D on which G acts isometrically by means of (possibly different) irreducible projective representations U and V respectively. In this case one can also replace $\mathcal{L}$ by $\overline{\mathcal{L}} := \{\bar{f}; f \in \mathcal{L}\}$ and replace V by $\overline{V}$, which is defined by $\overline{V}(f) := \overline{V(\bar{f})}$.

Notice that since $\text{span}\{K_w; w \in D\}$ is dense in $\mathcal{H}$, $\mathcal{A}_b$ is determined by its action on the kernel functions $K_w, w \in D$. We define

$$A_b(z,w) := \mathcal{A}_b\,(K_w)(z)/K_w(z) = \frac{\langle \mathcal{A}_b K_w, K_z\rangle}{\langle K_w, K_z\rangle}. \tag{2.33}$$

Then $\mathcal{A}_b$ is completely determined by the function $A_b(z,w)$. Notice that the function $A_b(z,z)$ is the *Berezin symbol* of $\mathcal{A}_b$, and it determines the function $A_b(z,w)$, since the latter is sesqui-holomorphic in (z,w), i.e. holomorphic in z and anti-holomorphic in w, see subsection 3.1. Also, the mapping $b \mapsto A_b$ (2.33) is G*-covariant*, i.e.

$$A_b(g(z), g(w)) = A_{b\circ g}(z,w), \quad \forall b \in \text{Dom}(\mathcal{A}). \tag{2.34}$$

Indeed, this follows easily by (1.17) and (2.32).

Conversely, if $b \mapsto A_b$ is a map from a G-invariant space $\mathcal{X}$ of functions on D into functions $A_b(z,w)$ on $D \times D$, which are sesqui-holomorphic and satisfy (2.34), then we can define a map $\mathcal{X} \ni b \mapsto \mathcal{A}_b \in Op(\mathcal{H})$ via $\mathcal{A}_b(K_w)(z) = K(z,w)A_b(z,w)$, and (2.32) will hold. Thus, *the invariant symbolic calculi are in one-to-one correspondence with the covariant maps into sesqui-holomorphic functions.* A similar statement is true for the version of the theory in which $\mathcal{A}_b : \mathcal{H} \to \mathcal{L}$.

Next, define an operator $\beta : \mathcal{X} = \text{Dom}(\mathcal{A}) \to \{\text{holomorphic functions on } D\}$ by

$$\beta(b)(z) = A_b(z,o) = \mathcal{A}_b(K_o)(z)/K_o(z). \tag{2.35}$$

Then β is K*-invariant*, i.e.

$$\beta(b \circ k) = \beta(b) \circ k, \qquad \forall\, k \in K.$$

Indeed, by (2.34) $\beta(b)(kz) = A_b(kz,o) = A_b(kz,ko) = A_{b\circ k}(z,o) = \beta(b\circ k)(z)$. Also, β determines A and $\mathcal{A}$ via

$$A_b(z,w) = \beta(b\circ g)(g^{-1}(z)) \quad \text{and} \quad \mathcal{A}_b(K_w)(z) = K(z,w)\,\beta(b\circ g)(g^{-1}(z)), \tag{2.36}$$

where $g \in G$ is any element for which $g(o) = w$. We therefore obtain

Corollary 2.1 (2.35) *and* (2.36) *establish a one-to-one correspondence between the invariant symbolic calculi* $b \mapsto \mathcal{A}_b$ *and the* K*-invariant maps* $b \mapsto \beta(b)$.

In what follows we shall assume that β is an *integral operator*:

$$\beta(b)(z) = \int_D F(z,w)\, b(w)\, d\mu_0(w),$$

where $F(z,w)$ is holomorphic in z and is K-invariant in the sense that

$$F(k(z), k(w)) = F(z,w), \quad \forall\, k \in K \;\; \forall\, z,w \in D. \tag{2.37}$$

This is indeed the case for the interesting calculi $\mathcal{A}$ on the weighted Fock and Bergman spaces. Next, for each $\eta \in D$ define an operator B_η on $\mathcal{H}$ via its action on the reproducing kernel functions

$$B_\eta(K_w)(z) = F\left(g^{-1}(z), g^{-1}(\eta)\right)\, K(z,w), \tag{2.38}$$

where, again, g is any element of G with $g(o) = w$. It is well-known and easy to prove that the kernel functions $\{K_w\}_{w\in D}$ are linearly independent. Hence, the K-invariance (2.37) shows that B_η is well-defined (independent of the choice of g) on span$\{K_w; w \in D\}$. Under mild assumptions on F (or β) B_η is also a closed operator. This property can be derived also on the basis of the properties of the adjoint $\mathcal{A}'$ of $\mathcal{A}$, which will be studied in the next subsection (see (2.49)).

Proposition 2.1 *For every $b \in Dom(\mathcal{A})$,*

$$\mathcal{A}_b = \int_D b(\eta)\, B_\eta \, d\mu_0(\eta), \tag{2.39}$$

where the integral converges in the weak operator topology on the dense subspace span$\{K_w; w \in D\}$.

Proof: It is enough to act on the reproducing kernel functions. Now (with $g \in G$, $g(o) = w$),

$$\begin{aligned}
\mathcal{A}_b(K_w)(z) &= A_b(z,w)\,K(z,w) = \beta(b\circ g)(g^{-1}(z))\;K(z,w)\\
&= \int_D F\left(g^{-1}(z),\xi\right)\; b(g(\xi))\; d\mu_0(\xi)\; K(z,w)\\
&= \int_D F\left(g^{-1}(z),g^{-1}(\eta)\right)\; b(\eta)\; d\mu_0(\eta)\; K(z,w)\\
&= \int_D b(\eta)\; B_\eta(K_w)(z)\; d\mu_0(\eta) = \left(\int_D b(\eta)\; B_\eta\; d\mu_0(\eta)\right)(K_w)(z).
\end{aligned}$$

Q.E.D

Proposition 2.2 *The map $D \ni \eta \mapsto B_\eta \in Op(\mathcal{H})$ is G-equivariant, namely*

$$U(g)^{-1}\,B_\eta\;U(g) = B_{g^{-1}(\eta)}, \qquad \forall\, g\in G, \qquad \forall \eta\in D. \tag{2.40}$$

Proof: Again, it is enough to act on the kernel functions $\{K_w\}_{w\in D}$. Now, for all $g \in G$,

$$\begin{aligned}
\left(U(g)^{-1}\;B_\eta\;U(g)\;K_w\right)(z) &= \overline{j(g,w)}\;\left(U(g^{-1})\;B_\eta\;K_{g(w)}\right)(z)\\
&= \overline{j(g,w)}\;j(g,z)\;(B_\eta K_{g(w)})(g(z))\\
&= j(g,z)\,\overline{j(g,w)}\,F((gh)^{-1}(g(z)),(gh)^{-1}(\eta))\,K(g(z),\,g(w))
\end{aligned}$$

where $h \in G$ is any element for which $h(o) = w$. Using (1.30) we see that the last expression is

$$F\left(h^{-1}(z),\,h^{-1}(g^{-1}(\eta))\right)\;K(z,w) = B_{g^{-1}(\eta)}(K_w)(z).$$

Q.E.D.

Definition 2.2 *A* covariant field of operators *on $\mathcal{H}$ is a family $\{B_\eta\}_{\eta\in D} \subseteq Op(\mathcal{H})$ satisfying* (2.40).

Corollary 2.2 *Given a covariant field of operators $\{B_\eta\}_{\eta\in D}$ on $\mathcal{H}$ define a map $b \mapsto \mathcal{A}_b$ via* (2.39)*. Then* (2.32) *holds. Thus, the invariant symbolic calculi $\mathcal{A}$ are in one-to-one correspondence with the covariant fields of operators $\{B_\eta\}_{\eta\in D}$ via* (2.39).

Proposition 2.3 *Let $\{B_\eta\}_{\eta\in D}$ be a covariant field of operators on $\mathcal{H}$.*
(i) The operator B_o is K-invariant, i.e.

$$U(k)B_oU(k)^{-1} = B_o, \quad \forall k \in K. \tag{2.41}$$

(ii) B_o determines all other operators B_η via

$$B_\eta = U(g)\ B_o\ U(g)^{-1}, \quad g \in G, \quad g(o) = \eta. \tag{2.42}$$

(iii) Given a K-invariant operator B on $\mathcal{H}$, define $B_o := B$, and let B_η be defined by (2.42). Then $\{B_\eta\}_{\eta\in D}$ are well-defined (independent of the choice of $g \in G$ for which $g(o) = \eta$) and form a covariant field of operators. Thus, (2.42) establishes a one-to-one correspondence between the covariant fields of operators on $\mathcal{H}$ and the K-invariant operators on $\mathcal{H}$.

Proof: (2.41) follows from (2.40): $U(k)B_o\,U(k)^{-1} = B_k(o) = B_o,\ \forall\ k \in K$. Also (2.42) follows from (2.40). Finally, (iii) follows from the fact that U is a projective representation. Q.E.D.

Corollary 2.3 *The invariant symbolic calculi $\mathcal{A}$ are in one-to-one correspondence with the K-invariant operators B on $\mathcal{H}$, $B \leftrightarrow \mathcal{A}^B$, via*

$$\mathcal{A}_b^B = \int_G b(g(o))\,U(g)\ B\ U(g)^{-1}\ dg = \int_D b(\xi)B_\xi\,d\mu_0.$$

Moreover, $\mathcal{A}^B$ is given also by the following formulas

$$\mathcal{A}_b^B = \int_D b(\eta)\,U(g_\eta)\ B\ U(g_\eta)^{-1}\ d\mu_0(\eta) = \int_D b(\eta)\,U(\varphi_\eta)\ B\ U(\varphi_\eta)\ d\mu_0(\eta),$$

where $g_\eta \in NA$ satisfies $g_\eta(o) = \eta$, and $\varphi_\eta \in G$ is the symmetry at the geodesic mid-point between o and η.

Remark: It is possible to develop our theory so that $\mathcal{A}$ will extend to a certain class of distributions. Then, the K-invariant operator B which determines $\mathcal{A}$ is given by $B = \mathcal{A}_{\delta_0}$, and the associated covariant field of operators is given by $B_\xi = \mathcal{A}_{\delta_\xi}$.

The next class of examples of invariant symbolic calculi, related to irreducible representations of K, is described in the setting of symmetric tube domains (cf. section 1.3). The minor notational changes needed for the weighted Fock spaces are obvious. Let $\{\mathcal{P}_{\mathbf{m}}\}_{\mathbf{m}\geq 0}$ be the irreducible K-invariant subspaces of $\mathcal{H}$. It is known that these subspaces are pairwise orthogonal and mutually K-inequivalent, and $\sum_{\mathbf{m}\geq 0} \mathcal{P}_{\mathbf{m}}$ is dense in $\mathcal{H}$ (see [Sch69]). Therefore Schur's lemma in representation theory implies that every K-invariant operator B on $\mathcal{H}$ leaves each $\mathcal{P}_{\mathbf{m}}$ invariant and $B|_{\mathcal{P}_{\mathbf{m}}} = b_{\mathbf{m}}\ I_{\mathcal{P}_{\mathbf{m}}}$. Thus

$$B = \sum_{\mathbf{m}\geq 0} b_{\mathbf{m}} P_{\mathbf{m}}$$

where $P_{\mathbf{m}}$ is the projection onto $\mathcal{P}_{\mathbf{m}}$ which annihilates all the spaces $\mathcal{P}_{\mathbf{n}}$ for $\mathbf{n} \neq \mathbf{m}$.

Corollary 2.4 *The invariant symbolic calculi are in one-to-one correspondence with families $\{b_{\mathbf{m}}\}_{\mathbf{m}\geq 0}$ of complex numbers, via*

$$\mathcal{A} \longleftrightarrow B = \sum_{\mathbf{m}\geq 0} b_{\mathbf{m}}\ P_{\mathbf{m}}.$$

Remark: B_o is bounded on $\mathcal{H}$ if and only if all the $\{B_\eta;\ \eta \in D\}$ are bounded, and $\|B_\eta\| = \|B_o\|,\ \forall\, \eta \in D$. Moreover, $B_o = \sum_{\mathbf{m}\geq 0} b_{\mathbf{m}} P_{\mathbf{m}}$ is bounded if and only if $\{b_{\mathbf{m}}\}_{\mathbf{m}\geq 0}$ is bounded.

Definition 2.3 $\mathcal{A}^{\mathbf{m}} = \mathcal{A}^{P_{\mathbf{m}}}$, *i.e.* $\mathcal{A}^{\mathbf{m}}$ *is the invariant symbolic calculus determined by the K-invariant operator $B = P_{\mathbf{m}}$. Let $A^{\mathbf{m}}$, $F^{\mathbf{m}}$, and $\beta^{\mathbf{m}}$ be the maps associated with $\mathcal{A}^{\mathbf{m}}$ in the manner described above.*

The importance of the $\mathcal{A}^{\mathbf{m}}$'s is exhibited in the following simple fact.

Proposition 2.4 *Given a K-invariant operator $T = \sum_{\mathbf{m}\geq 0} t_{\mathbf{m}}\ P_{\mathbf{m}}$ on $\mathcal{H}$, the corresponding invariant symbolic calculus is given by $\mathcal{A}_b = \mathcal{A}_b^T = \sum_{\mathbf{m}\geq 0} t_{\mathbf{m}}\ \mathcal{A}_b^{\mathbf{m}}$.*

Definition 2.4 *The covariant field of operators associated with $P_{\mathbf{m}}$ is*

$$P_{\mathbf{m},\eta} = U(g)\ P_{\mathbf{m}}\ U(g)^{-1}, \quad g \in G,\ \ g(o) = \eta.$$

Definition 2.5 *The reproducing kernel of $\mathcal{P}_{\mathbf{m}}$ with respect to the inner product of $\mathcal{H}$ is denoted by $K_{\mathbf{m}}(z,w) = K_{\mathbf{m}}^{\mathcal{H}}(z,w)$.*

Thus

$$P_{\mathbf{m}}(f)(z) = \langle f,\ K_{\mathbf{m}}(\cdot, z)\rangle_{\mathcal{H}}.$$

Lemma 2.1 *Let $\mathbf{m} \geq 0$ and $\eta \in D$. Then for every $g \in G$ with $g^{-1}(o) = \eta$ we have*

$$P_{\mathbf{m},\eta}(K_w)(z) = j(g,z)\ K_{\mathbf{m}}(g(z), g(w))\ \overline{j(g,w)} \tag{2.43}$$

$$A_b^{\mathbf{m}}(z,w) = K(z,w)^{-1} \int_D b(\eta)\ K_{\mathbf{m}}(g_\eta^{-1}(z), g_\eta^{-1}(w))\ j(g_\eta^{-1}, z)\ \overline{j(g_\eta^{-1}, w)}\ d\mu_0(\eta) \tag{2.44}$$

where g_η is the unique element of NA for which $g_\eta(o) = \eta$. In particular,

$$\beta^{\mathbf{m}}(b)(z) = K(z,o)^{-1} \int_D b(\eta)\ K_{\mathbf{m}}(g_\eta^{-1}(z), g_\eta^{-1}(o))\ j(g_\eta^{-1}, z)\ \overline{j(g_\eta^{-1}, o)}\ d\mu_0(\eta) \tag{2.45}$$

and

$$F^{\mathbf{m}}(z,\eta) = K(z,o)^{-1} K_{\mathbf{m}}(g_\eta^{-1}(z), g_\eta^{-1}(o))\ j(g_\eta^{-1}, z)\ \overline{j(g_\eta^{-1}, o)}. \tag{2.46}$$

Proof: By definition of $P_{\mathbf{m},\eta}$ we have

$$\begin{aligned} P_{\mathbf{m},\eta}\,(K_w)(z) &= \big(U(g)^{-1}\,P_{\mathbf{m}}\,U(g)\;K_w\big)\,(z)\\ &= P_{\mathbf{m}}\,(U(g)\,K_w)\;(g(z))\;j(g,z)\\ &= \langle U(g)\,K_w,\;K_{\mathbf{m}}(\cdot,g(z))\rangle_{\mathcal{H}}\;j(g,z)\\ &= \langle K_w, K_{\mathbf{m}}(g(\cdot),g(z))\rangle_{\mathcal{H}}\;j(g,z)\;\overline{j(g,w)}\\ &= K_{\mathbf{m}}(g(z),g(w))\;j(g,z)\;\overline{j(g,w)}. \end{aligned}$$

This establishes (2.43). Next,

$$\begin{aligned} \mathcal{A}_b^{\mathbf{m}}(K_w)(z) &= \int\limits_D b(\eta)\;P_{\mathbf{m},\eta}(K_w)(z)\;d\mu_0(\eta)\\ &= \int\limits_D b(\eta)\;K_{\mathbf{m}}\,(g_\eta^{-1}(z),g_\eta^{-1}(w))\;j(g_\eta^{-1},z)\;j(g_\eta^{-1},w)\;d\mu_0(\eta). \end{aligned}$$

This implies (2.44). Finally, (2.45) and (2.46) are direct consequences of (2.44). Q.E.D.

Remark: In (2.44), (2.45) and (2.46) we can replace g_η by any other $g\in G$ for which $g(o)=\eta$.

In what follows we describe the basic examples of invariant symbolic calculi.

Example 2.1 The most important symbolic calculus is the *Toeplitz calculus* (called also "Toeplitz quantization", or "Toeplitz-Berezin quantization"). Let us consider the case where $\mathcal{H}$ is the Bergman space $\mathcal{H}=L^2_a(D,\mu):=L^2(D,\mu)\cap\{\text{holomorphic functions on }\;D\}$, and μ is a measure on D satisfying (1.20). Let $P:L^2(D,\mu)\to\mathcal{H}$ be the orthogonal projection. The *Toeplitz operator* with an active (strong) symbol $b\in L^\infty(D,\mu)$ is the operator $\mathcal{T}_b:\mathcal{H}\to\mathcal{H}$ defined by

$$(\mathcal{T}_b f)(z):=P(bf)(z)=\int_D b(w)f(w)K(z,w)\,d\mu(w),\quad f\in\mathcal{H},\;\; z\in D.$$

It is well known that (2.32) holds with $\mathcal{A}=\mathcal{T}$.

Example 2.2 Another important invariant symbolic calculus, the *Weyl calculus* $\mathcal{W}$, is defined in the general setting via

$$\mathcal{W}_b:=\int_D b(\eta)\,U(s_\eta)\,d\mu_0(\eta),$$

where $s_\eta\in G$ is the symmetry at η and μ_0 is the G-invariant measure on D. Namely, the domain of definition $\mathrm{Dom}(\mathcal{W})$ of $\mathcal{W}$ consists of all measurable functions b on D for which the integral

$$\mathcal{W}_b(f)(z):=\int_D b(\eta)\,(U(s_\eta)f)(z)\,d\mu_0(\eta)$$

converges weakly in $\mathcal{H}$ for all $f\in\mathcal{H}$. Using (1.1), (1.14) and (1.16) one obtains

$$U(g)\,U(s_\eta)\,U(g)^{-1}=U(s_{g(\eta)}),\quad \forall g\in G,\;\;\forall\eta\in D,$$

and this implies that (2.32) holds with $\mathcal{A}=\mathcal{W}$.

2.2 The adjoint map and the link transform

The *adjoint* of an invariant symbolic calculus $b \mapsto \mathcal{A}_b$ is the map $\mathcal{A}'$ from $Op(\mathcal{H})$, the space of closed operators on $\mathcal{H}$, to functions on D is defined via

$$\mathrm{Dom}(\mathcal{A}') := \{T \in Op(\mathcal{H});\ T\mathcal{A}_b^* \in S_1(\mathcal{H})\ \ \forall b \in \mathrm{Dom}(\mathcal{A})\},$$

where $S_1(\mathcal{H})$ is the space of *trace class* operators on $\mathcal{H}$, and

$$\langle \mathcal{A}'(T), b\rangle_{L^2(D,\mu_0)} = \langle T, \mathcal{A}_b\rangle_{S_2} = \mathrm{trace}(T\,\mathcal{A}_b^*) \tag{2.47}$$

for all $T \in \mathrm{Dom}(\mathcal{A}')$ and for all $b \in \mathrm{Dom}(\mathcal{A})$. Here S_2 is the *Hilbert-Schmidt class.* The function $\mathcal{A}'(T)$ is called the *passive* (or, weak) *symbol* of T.

Proposition 2.5 (i) *For every $T \in Dom(\mathcal{A}')$,*

$$\mathcal{A}'\left(U(g)\ T\ U(g)^{-1}\right) = \mathcal{A}'(T) \circ g^{-1}, \qquad \forall\ g \in G. \tag{2.48}$$

(ii) *For every $T \in Dom(\mathcal{A}')$,*

$$\mathcal{A}'(T)(\eta) = \langle T, B_\eta\rangle_{S_2}, \qquad \forall \eta \in D. \tag{2.49}$$

Proof: (i) is a consequence of (2.32) and (2.47). From (2.39) we obtain, formally,

$$\langle \mathcal{A}'(T), b\rangle_{L^2(\mu_0)} = \langle T, \int_D b(\eta)\ B_\eta\ d\mu_0(\eta)\rangle_{S_2} = \int_D \langle T, B_\eta\rangle_{S_2}\overline{b(\eta)}\, d\mu_0(\eta),$$

and this yields (2.49). Q.E.D.

Example 2.3 (i) For the Toeplitz calculus $\mathcal{T}$ (see Example 2.1) we have $B_z = k_z \otimes k_z$, where $k_z := K_z/\|K_z\|$. Hence

$$\mathcal{T}'(T)(z) = \langle T\ k_z, k_z\rangle_{\mathcal{H}}$$

is the *Berezin symbol* of T.

(ii) For the Weyl calculus $\mathcal{W}$ (see Example 2.2) we have

$$\mathcal{W}'(T)(\eta) = \langle T, U(s_\eta)\rangle_{S_2},$$

where $s_\eta \in G$ is the symmetry at $\eta \in D$.

Remark: (2.48) implies that the function $\mathcal{A}'(I)$ satisfies $\mathcal{A}'(I) \circ g = \mathcal{A}'(I)$ for every $g \in G$. Hence $\mathcal{A}'(I)$ is a constant function. We therefore assume without loss of generality (by modifying the definition of $\mathcal{A}'$ if necessary) that

$$\mathcal{A}'(I)(z) = 1, \quad \forall z \in D.$$

We now associate with an invariant symbolic calculus $\mathcal{A}$ two linear transformations, namely

$$\mathcal{B} := \mathcal{A}'\,\mathcal{A} \quad \text{and} \quad \mathcal{Q} := \mathcal{A}\,\mathcal{A}',$$

acting on functions on D, and on operators on $\mathcal{H}$ respectively. The operator $\mathcal{B}$ is called the *link transform* associated with $\mathcal{A}$, because it maps the active symbol b of $\mathcal{A}_b$ to the passive symbol $\mathcal{B}\, b = \mathcal{A}'\,(\mathcal{A}_b)$ of $\mathcal{A}_b$. We call $\mathcal{Q}$ the corresponding "co-transform". It maps an operator S to the operator $\mathcal{Q}(S)$ whose active symbol is the passive symbol of S.

Example 2.4 The link transform associated with the Toeplitz calculus $\mathcal{T}$ on the Bergman space $\mathcal{H} = L^2_a(D,\mu)$ with respect to a K-invariant measure μ is the *Berezin transform* $\mathcal{B} = \mathcal{T}'\mathcal{T}$ associated with μ:

$$\mathcal{B}(b)(z) = \langle b\, k_z, k_z\rangle_{\mathcal{H}} = \int_D b(w)\, \mathcal{K}(z,w)\, d\mu_0(w),$$

where $d\mu_0(w) := K(w,w)\, d\mu(w)$ is the (properly normalized) G-invariant measure on D and

$$\mathcal{K}(z,w) = \frac{|K(z,w)|^2}{K(z,z)\ K(w,w)}$$

is a G-*invariant* kernel, i.e.: $\mathcal{K}(g(z),g(w)) = \mathcal{K}(z,w)$ for all $g \in G$. This exhibits $\mathcal{B}$ as the operator of *convolution with* μ:

$$(\mathcal{B}b)(z) = \int_D b \circ g\ d\mu, \quad g \in G, \ \ g(o) = z. \tag{2.50}$$

Also, in the case of the Toeplitz calculus the co-transform $\mathcal{Q} = \mathcal{T}\mathcal{T}'$ is given by

$$\mathcal{Q}(S) = \int_D \langle S\ k_\eta, k_\eta\rangle_{\mathcal{H}}\ \ k_\eta \otimes k_\eta\ \ d\mu_0(\eta).$$

The link transforms associated with the Weyl calculus $\mathcal{W}$ are given formally by

$$\mathcal{W}'\mathcal{W}(b)(\eta) = \int_D b(\xi)\langle U(s_\xi), U(s_\eta)\rangle_{S_2}\ d\mu_0(\xi)$$

and

$$\mathcal{W}\mathcal{W}'(T) = \int_D \langle T, U(s_\xi)\rangle_{S_2}\ U(s_\xi)\ d\mu_0(\xi).$$

Proposition 2.6 *Let $\mathcal{A}$ be an invariant symbolic calculus with adjoint $\mathcal{A}'$.*
(i) *The link transform $\mathcal{B} = \mathcal{A}'\,\mathcal{A}$ is given by*

$$(\mathcal{B}\, b)(\eta) = \int_D b(\xi)\ \langle B_\xi, B_\eta\rangle_{S_2}\ d\mu_0(\xi). \tag{2.51}$$

(ii) *The co-transform $\mathcal{Q} = \mathcal{A}\,\mathcal{A}'$ is given by*

$$\mathcal{Q}(S) = \int_D \langle S, B_\eta\rangle_{S_2}\, B_\eta\ d\mu_0(\eta). \tag{2.52}$$

Proof: (i) Using (2.39) and (2.49) we obtain

$$\begin{aligned}(\mathcal{B}b)(\eta) &= \mathcal{A}'\,(\mathcal{A}_b)(\eta) = \langle \mathcal{A}_b, B_\eta\rangle_{S_2} \\ &= \langle \int_D b(\xi)\ B_\xi\ d\mu_0(\xi), B_\eta\rangle_{S_2} = \int_D \langle B_\xi, B_\eta\rangle_{S_2}\ d\mu_0(\xi).\end{aligned}$$

(ii) Formula (2.52) is a consequence of (2.39) and (2.49). Q.E.D.

Corollary 2.5 *For $b_1, b_2 \in \mathrm{Dom}\,(\mathcal{A})$,*

$$\langle \mathcal{A}_{b_1}, \mathcal{A}_{b_2}\rangle_{S_2} = \iint_{D\times D} b_1(\xi)\ \overline{b_2(\eta)}\ \langle B_\xi, B_\eta\rangle_{S_2}\ d\mu_0(\xi)\ d\mu_0(\eta).$$

Proof: Using (2.51) we obtain

$$\begin{aligned}\langle \mathcal{A}_{b_1}, \mathcal{A}_{b_2}\rangle_{S_2} &= \langle \mathcal{B}\, b_1, b_2\rangle_{L^2(\mu_0)} = \int_D (\mathcal{B}b_1(\xi)\ \overline{b_2(\eta)}\ d\mu_0(\eta)\\ &= \iint_{D\times D} b_1(\xi)\ \overline{b_2(\eta)}\ \langle B_\xi, B_\eta\rangle_{S_2}\ d\mu_0(\xi)\ d\mu_0(\eta).\end{aligned}$$

Q.E.D.

Proposition 2.7 (i) *The link transform $\mathcal{B} = \mathcal{A}'\mathcal{A}$ is G-invariant, i.e.*

$$\mathcal{B}(b\circ g) = (\mathcal{B}\,b)\circ g, \quad \forall\ g \in G.$$

(ii) *The co-transform $\mathcal{Q} = \mathcal{A}\,\mathcal{A}'$ is G-invariant in the sense that*

$$U(g)\ \mathcal{Q}(S)\ U(g)^{-1} = \mathcal{Q}\left(U(g)\,S\,U(g)^{-1}\right), \quad \forall\ g \in G.$$

Proof: This follows from (2.32) and (2.48). Q.E.D.

Remark: If $1 \in \mathrm{Dom}(\mathcal{A})$ then $1 \in Dom(\mathcal{A}'\mathcal{A})$ as well, and

$$\mathcal{A}'\mathcal{A}(1) = 1.$$

Indeed, this follows from the facts that $\mathcal{A}_1 = I$ and $\mathcal{A}'(I) = 1$.

Lemma 2.2 *Let $\xi, \eta \in D$. Then*

$$\langle B_\xi, B_\eta\rangle_{S_2} = \int_D \int_D F(g^{-1}(z), g^{-1}(\xi))\, F(g^{-1}(z), g^{-1}(\eta))\, |K(z,w)|^2\ d\mu(z)\ d\mu(w) \tag{2.53}$$

where $g \in G$ in the inner integral is an arbitrary element for which $g(o) = w$.

Proof: It is a well-known fact that for every operator T on $\mathcal{H} = L^2_a(D,\mu)$ for which the trace $tr(T)$ is well defined (i.e., $T \in S_1$ or $T \geq 0$),

$$tr(T) = \int_D \langle T(K_w), K_w\rangle_{\mathcal{H}}\, d\mu(w).$$

Therefore, using (2.38) we obtain

$$\begin{aligned}\langle B_\xi, B_\eta\rangle_{S_2} &= tr(B^*_\eta\, B_\xi) = \int_D \langle B_\xi(K_w), B_\eta(K_w)\rangle_{\mathcal{H}}\ d\mu(w)\\ &= \int_D\int_D F(g^{-1}(z), g^{-1}(\xi))\, \overline{F(g^{-1}(z), g^{-1}(\eta))}\, |K(z,w)|^2\, d\mu(z)\, d\mu(w).\end{aligned}$$

Q.E.D.

Definition 2.6 *Given two K-invariant operators T and S on $\mathcal{H}$, let $\mathcal{A}^T$, $\mathcal{A}^S$ be the associated invariant symbolic calculi. The corresponding mixed link transform and co-transform are*

$$\mathcal{B}^{T,S} := (\mathcal{A}^S)' \,\mathcal{A}^T \qquad \text{and} \qquad \mathcal{Q}^{T,S} := \mathcal{A}^T \,(\mathcal{A}^S)' \tag{2.54}$$

respectively.

It is clear that these two transforms are G-invariant, i.e.

$$\mathcal{B}^{T,S}(b \circ g) = (\mathcal{B}^{T,S}\, b) \circ g, \quad \forall\, b \in \mathrm{Dom}(\mathcal{B}^{T,S}), \quad \forall\, g \in G$$

and

$$U(g)\; \mathcal{Q}^{T,S}(X)\; U(g)^{-1} = \mathcal{Q}^{T,S}\left(U(g)\, X\, U(g)^{-1}\right), \quad \forall g \in G, \ \forall X \in \mathrm{Dom}(\mathcal{Q}^{T,S}).$$

Our results extend to the context of the mixed link transforms and co-transforms.

2.3 The reflection ψ and the associated involution of $\mathfrak{A}^{*\mathbb{C}}$

In this subsection we study in detail the real-analytic reflection ψ of D at o mentioned above (1.3) and its connection to a natural involution on $\mathfrak{A}^{*\mathbb{C}}$. Recall that for any $z \in D$ there exists a unique element $g_z \in NA \subset G$ for which $g_z(o) = z$. The map $\psi : D \to D$ is defined by

$$\psi(z) = g_z^{-1}(o). \tag{2.55}$$

Proposition 2.8 *ψ is a real-analytic diffeomorphism of D satisfying $\psi(\psi(z)) = z$ for all $z \in D$. The unique fixed point of ψ is o.*

Proof: Let $w = \psi(z) = g_z^{-1}(o)$. Then $g_w(o) = w = g_z^{-1}(o)$ hence $g_w = g_z^{-1}$. Thus $\psi(w) = g_z(o) = z$, i.e. $\psi(\psi(z)) = z$. The fact that ψ is a real-analytic diffeomorphism follows from the fact that $NA \ni g \mapsto g(o) \in D$ is a real-analytic diffeomorphism. Finally, $\psi(z) = z$ if and only if $g_z \circ g_z = 1_G$. Since $g_z \in NA$, this is equivalent to $g_z = 1_G$, and so $z = g_z(o) = 1_G(o) = o$. Q.E.D.

In view of Proposition 2.8 we call ψ a *reflection* of D at o. Unless $D = \mathbb{C}^d$, ψ is not holomorphic, and thus - not a member of G.

Lemma 2.3 *There exists an involution $\underline{\lambda} \mapsto \underline{\lambda}^*$ on the Lie algebra $\mathfrak{A}^{*\mathbb{C}} \equiv \mathbb{C}^r$ such that*

$$\int_D \overline{e_{\underline{\lambda}}(\xi)}\; f(\psi(\xi))\, d\mu_0(\xi) = \int_D e_{\underline{\lambda}^*}(\xi)\; f(\xi)\; d\mu_0(\xi) = \tilde{f}(\underline{\lambda}^*)$$

holds for all admissible functions f on D and $\underline{\lambda} \in \mathbb{C}^r$.

Proof: In the case of the Fock space on $\mathbb{C}^d$ (see subsection 1.2 above) we have $g_\xi(z) = z + \xi$ and thus $\psi(\xi) = (g_\xi)^{-1}(0) = -\xi$. Hence, for $a, b \in \mathbb{C}^d$ with $\langle a, b\rangle = \lambda$ one can take $e_\lambda(z) = e_{a,b}(z) = \exp(\langle a, z\rangle + \langle z, b\rangle)$. Therefore one obtains

$$\int_{\mathbb{C}^d} \overline{e_{a,b}(\xi)}\; f(\psi(\xi))\; d\mu_0(\xi) = \int_{\mathbb{C}^d} e_{-b,-a}(\xi)\; f(\xi)\; d\mu_0(\xi). \tag{2.56}$$

Since $\langle -b, -a \rangle = \lambda$ the involution $*$ satisfying (2.56) is

$$\lambda^* := \bar{\lambda}. \tag{2.57}$$

In the case of a symmetric tube domain $T(\Omega) \subset \mathbb{C}^d$ we shall prove in Lemma 2.4 below that

$$(\widetilde{f \circ \psi})(\underline{\lambda}) = \tilde{f}(-\underline{\lambda}). \tag{2.58}$$

Since $\overline{e_{\underline{\lambda}}(z)} = e_{\overline{\underline{\lambda}}}(z)$ in this case, we obtain

$$\int\limits_{T(\Omega)} \overline{e_{\underline{\lambda}}(\xi)}\ f(\psi(\xi))\ d\mu_0(\xi) = \int\limits_{T(\Omega)} e_{-\overline{\underline{\lambda}}}(\xi)\ f(\xi)\ d\mu_0(\xi).$$

Hence Lemma 2.3 holds with the involution

$$\underline{\lambda}^* = -\overline{\underline{\lambda}}\,. \tag{2.59}$$

Q.E.D

It remains to prove (2.58) in the setting of symmetric tube domains.

Lemma 2.4 *In the context of symmetric tube domains $T(\Omega)$, the map ψ satisfies*

$$Det_{\mathbb{R}}((d\psi)(z)) = N_{2\underline{\rho}-2p}(\tau(z)).$$

Proof: The *evaluation map*

$$\varepsilon : NA \to T(\Omega),\ \varepsilon(g) = g(ie)$$

is a real-analytic diffeomorphism whose differential

$$d\varepsilon(\mathbf{1}) : \mathfrak{a} \oplus \mathfrak{n} = T_{\mathbf{1}}(NA) \to Z = T_{ie}(T(\Omega))$$

is also an evaluation map

$$d\varepsilon(\mathbf{1})(X) = X(ie), \quad \forall\ X \in \mathfrak{a} \oplus \mathfrak{n}$$

which is a real-linear isomorphism of $\mathfrak{a} \oplus \mathfrak{n}$ onto Z. For every $g \in NA$ consider the inner automorphism

$$c_g(h) = ghg^{-1}, \quad h \in G$$

and its differential

$$Ad(g) = dc_g(\mathbf{1}) : \mathfrak{a} \oplus \mathfrak{n} \to \mathfrak{a} \oplus \mathfrak{n}\ .$$

Let $j : NA \to NA$ be the inversion map $j(h) = h^{-1}$. We claim that for every $g \in NA$ we have

$$g^{-1} \circ \varepsilon \circ j \circ c_g = \psi \circ g \circ \varepsilon, \tag{2.60}$$

namely the following diagram is commutative

$$\begin{array}{ccccccc} NA & \xrightarrow{c_g} & NA & \xrightarrow{j} & NA & \xrightarrow{\varepsilon} & T(\Omega) \\ \varepsilon \downarrow & & & & & & \downarrow g^{-1} \\ T(\Omega) & \xrightarrow{g} & & T(\Omega) & & \xrightarrow{\psi} & T(\Omega) \end{array} \tag{2.61}$$

Indeed, for every $h \in NA$

$$\begin{aligned}(g^{-1} \circ \varepsilon \circ j \circ c_g)(h) &= g^{-1}(((ghg^{-1})^{-1})(ie)) \\ &= g^{-1}(g(h^{-1}(g^{-1}(ie)))) = h^{-1}(g^{-1}(ie)).\end{aligned}$$

Also,

$$(\psi \circ g \circ \varepsilon)(h) = \psi(g(h(ie))) = (gh)^{-1}(ie) = h^{-1}(g^{-1}(ie)),$$

and (2.60) is proved. Since $T_w(T(\Omega)) = Z$ for all $w \in T(\Omega)$ and $dj(\mathbf{1}) = -I$, we obtain from (2.61) by taking differentials the following commutative diagram

$$\begin{array}{ccccccc} \mathfrak{a} \oplus \mathfrak{n} & \xrightarrow{Ad(g)} & \mathfrak{a} \oplus \mathfrak{n} & \xrightarrow{-I} & \mathfrak{a} \oplus \mathfrak{n} & \xrightarrow{d\varepsilon(\mathbf{1})} & Z \\ d\varepsilon(\mathbf{1}) \downarrow & & & & & & \downarrow d(g^{-1})(ie) \\ Z & \xrightarrow{dg(ie)} & & Z & & \xrightarrow{d\psi(z)} & Z, \end{array} \tag{2.62}$$

where $z := g(ie)$. Identifying $\mathfrak{a} \oplus \mathfrak{n}$ and Z via the evaluation map $d\varepsilon(\mathbf{1})$, we obtain from (2.62) by taking determinants

$$Det_{\mathbb{R}}(Ad(g))Det_{\mathbb{R}}(d(g^{-1})(ie)) = Det_{\mathbb{R}}(dg(ie))Det_{\mathbb{R}}(d\psi(z)).$$

Now the well-known modulus function of NA, determined in a general setting in [AU99], yields

$$Det_{\mathbb{R}}\, Ad(g) = N_{2\underline{\rho}}(\tau(z)).$$

Since $Det_{\mathbb{R}}(dg(ie)) = N(\tau(z))^p$ and $Det_{\mathbb{R}}(dg^{-1}(ie)) = N(\tau(z)^{-1})^p = N(\tau(z))^{-p}$, we obtain

$$Det_{\mathbb{R}}(d\psi(z)) = \frac{Det_{\mathbb{R}}(Ad(g))}{N(\tau(z))^{2p}} = N_{2\underline{\rho}-2p}(\tau(z)).$$

Q.E.D.

Corollary 2.6 *The G-invariant measure μ_0 on $T(\Omega)$ transforms under ψ according to the rule*

$$d\mu_0(\psi(z)) = N_{2\underline{\rho}}(\tau(z))\, d\mu_0(z).$$

Proof: (1.27) shows that

$$N(\tau(\psi(z))) = N(\tau(g_z^{-1}(ie))) = N(\tau(g_z(ie)))^{-1} = N(\tau(z))^{-1}.$$

Therefore we obtain

$$\begin{aligned}d\mu_0(\psi(z)) &= N(\tau(\psi(z)))^{-p} dm(\psi(z)) \\ &= N(\tau(z))^p\, N_{2\underline{\rho}-2p}(\tau(z))\, dm(z) = N_{2\underline{\rho}}(\tau(z))\, d\mu_0(z).\end{aligned}$$

Q.E.D.

Corollary 2.7 *For every $\underline{\lambda} \in \mathbb{C}^r$ and $z \in T(\Omega)$*

$$e_{\underline{\lambda}}(\psi(z)) = e_{-\underline{\lambda}-\underline{\rho}}(z).$$

Proof: For every $\underline{\alpha} \in \mathbb{C}^r$ and $g \in NA$ we get by (1.27)

$$1 = N_{\underline{\alpha}}(\tau(ie)) = N_{\underline{\alpha}}(\tau(g(g^{-1}(ie)))) = N_{\underline{\alpha}}(\tau(g(ie)))\, N_{\underline{\alpha}}(\tau(g^{-1}(ie))).$$

Thus

$$N_{\underline{\alpha}}(\tau(g^{-1}(ie))) = \frac{1}{N_{\underline{\alpha}}(\tau(g(ie)))} = N_{-\underline{\alpha}}(\tau(g(ie))).$$

Putting $z = g(ie)$ and using (1.28), this implies

$$e_{\underline{\lambda}}(\psi(z)) = N_{\underline{\lambda}+\underline{\rho}}(\tau(g^{-1}(ie))) = N_{-\underline{\lambda}-\underline{\rho}}(\tau(z)) = e_{-\underline{\lambda}-2\underline{\rho}}(z).$$

Q.E.D.

Corollary 2.8 *Let f be a locally integrable function on $T(\Omega)$ and $\underline{\lambda} \in \mathbb{C}^r$. Then $\underline{\lambda} \in Dom\,(\widetilde{f \circ \psi})$ if and only if $-\underline{\lambda} \in Dom\,(\tilde{f})$ and*

$$\widetilde{f \circ \psi}\,(\underline{\lambda}) = \tilde{f}(-\underline{\lambda}).$$

Proof:

$$\begin{aligned}(\widetilde{f \circ \psi})\,(\underline{\lambda}) &= \int_{T(\Omega)} f(\psi(z))\, e_{\underline{\lambda}}(z)\, d\mu_0(z) = \int_{T(\Omega)} f(w)\, e_{\underline{\lambda}}(\psi(w))\, d\mu_0(\psi(w)) \\ &= \int_{T(\Omega)} f(w)\, e_{-\underline{\lambda}-2\underline{\rho}}(w)\ N_{2\underline{\rho}}(\tau(w))\, d\mu_0(w) \\ &= \int_{T(\Omega)} f(w)\, e_{-\underline{\lambda}}(w)\, d\mu_0(w) = \tilde{f}(-\underline{\lambda}).\end{aligned}$$

Q.E.D.

2.4 Eigenvalues of the link transforms

Let $\mathcal{A}$ be an invariant symbolic calculus on $\mathcal{H} = L^2_a(D,\mu)$, and let $\beta, F, \mathcal{A}'$ and $\mathcal{B}$ be as in previous sections. Recall that the "exponential functions" $e_{\underline{\lambda}}$, $\underline{\lambda} \in \mathbb{C}^r$, are joint eigen-functions of the G-invariant operators on D, and that the K-averages of the $e_{\underline{\lambda}}$'s are the spherical functions $\phi_{\underline{\lambda}}(z) = \int_K e_{\underline{\lambda}}(k(z))\, dk$. In particular, if $\mathcal{B}$ is a link transform then for every $e_{\underline{\lambda}} \in \mathrm{Dom}\,(\mathcal{B})$

$$\mathcal{B}(e_{\underline{\lambda}}) = \tilde{\mathcal{B}}(\underline{\lambda})\ e_{\underline{\lambda}}, \quad \text{where } \tilde{\mathcal{B}}(\underline{\lambda}) = \mathcal{B}(e_{\underline{\lambda}})(o).$$

In what follows we shall use the notation

$${}_zF(w) := F(z,w), \quad \forall z, w \in D$$

as well as the notion of the transform $\tilde{f}$ of functions f on D (see (1.11)).

Proposition 2.9 *For $e_{\underline{\lambda}} \in Dom\,(\mathcal{B})$,*

$$\tilde{\mathcal{B}}(\underline{\lambda}) = \int_D {}_z\tilde{F}(\underline{\lambda})(\widetilde{{}_zF \circ \psi})(\underline{\lambda})\, |K(z,o)|^2\, d\mu(z) \tag{2.63}$$

where ${}_z\tilde{F}(\underline{\lambda})$ and $(\widetilde{{}_zF \circ \psi})(\underline{\lambda})$ are defined via (1.11).

Proof: Using (2.51) with $b = e_{\underline{\lambda}}$ and $\eta = o$, as well as (2.53) we find

$$\tilde{\mathcal{B}}(\underline{\lambda}) = \int_D e_{\underline{\lambda}}(\xi)\ \langle B_\xi, B_o\rangle_{S_2}\ d\mu_0(\xi)$$

$$= \int_D e_{\underline{\lambda}}(\xi) \left\{ \int_D \left(\int_D F(g_w^{-1}(z), g_w^{-1}(\xi))\, \overline{F(g_w^{-1}(z), g_w^{-1}(o))} \right.\right.$$

$$\left.\left. |K(z,w)|^2\ d\mu(z) \right)\ d\mu(w) \right\}\ d\mu_0(\xi)\ .$$

Letting $u = g_w^{-1}(z)$, then

$$K(z,w) = K(g_w(u), g_w(o)) = K(u,o)/j(g_w,u)\ \overline{j(g_w,o)}\ .$$

Using (1.20) and (1.21) with g_w, we obtain

$$\tilde{\mathcal{B}}(\underline{\lambda}) = \int_D e_{\underline{\lambda}}(\xi) \int_D \left[\int_D {}_uF(g_w^{-1}(\xi))\, \overline{{}_uF(g_w^{-1}(o))}\ |K(u,o)|^2\ d\mu(u) \right] d\mu_0(w)\, d\mu_0(\xi).$$

Interchanging the order of integration, we obtain

$$\tilde{\mathcal{B}}(\underline{\lambda}) = \int_D |K(u,o)|^2\ d\mu(u) \int_D d\mu_0(w) \left[\int_D {}_uF(g_w^{-1}(\xi))\, e_{\underline{\lambda}}(\xi)\, d\mu_0(\xi) \right] \overline{{}_uF(\psi(w))}.$$

The substitution $\eta = g_w^{-1}(\xi)$ and the fact that $e_{\underline{\lambda}}(\xi) = e_{\underline{\lambda}}(g_w(\eta)) = e_{\underline{\lambda}}(w)\ e_{\underline{\lambda}}(\eta)$ lead to

$$\tilde{\mathcal{B}}(\underline{\lambda}) = \int_D |K(u,o)|^2\ d\mu(u) \left[\int_D d\mu_0(w)\ \overline{{}_uF(\psi(w))}\ e_{\underline{\lambda}}(w) \right] \int_D {}_uF(\eta)\ e_{\underline{\lambda}}(\eta)\ d\mu_0(\eta)$$

$$= \int_D \widetilde{{}_uF}(\underline{\lambda})\ \overline{\widetilde{{}_uF \circ \psi}}(\underline{\lambda})\ |K(u,o)|^2\ d\mu(u).$$

Q.E.D.

Formula (2.63) can be extended to the context of the link transform $\mathcal{B}^{T,S} = (\mathcal{A}^S)'\,\mathcal{A}^T$, see (2.54). Indeed, if F^T, F^S are the F-functions associated with $\mathcal{A}^T$ and $\mathcal{A}^S$ respectively then the proof of Proposition 2.9 yields also the following result.

Proposition 2.10 *The eigenvalues of the link transform $\mathcal{B}^{T,S}$ are given by*

$$\widetilde{\mathcal{B}^{T,S}}(\underline{\lambda}) = \int_D \widetilde{{}_zF^T}(\underline{\lambda})\ \overline{\widetilde{{}_zF^S \circ \psi}}(\underline{\lambda})\ |K(z,o)|^2\ d\mu(z).$$

Notice that, by definition,

$$\widetilde{{}_zF^T}(\underline{\lambda}) = \int_D e_{\underline{\lambda}}(\xi)\ F^T(z,\xi)\ d\mu_0(\xi) = \beta^T(e_{\underline{\lambda}})(z) = A^T_{e_{\underline{\lambda}}}(z,o) = A^T_{e_{\underline{\lambda}}}(K_o)(z)/K_o(z).$$

Also

$$(\widetilde{{}_zF^S \circ \psi})(\underline{\lambda}) = \int_D \overline{e_{\underline{\lambda}}(\xi)\; F(z,\psi(\xi))\, d\mu_0(\xi)}. \tag{2.64}$$

Using Lemma 2.3, (2.64) can be written as

$$\left(\widetilde{\overline{{}_zF^S \circ \psi}}\right)(\underline{\lambda}) = \overline{\int_D e_{\underline{\lambda}^*}(\xi)\; F^S(z,\xi)\; d\mu_0(\xi)} = \overline{\mathcal{A}^S_{e_{\underline{\lambda}^*}}(K_o)(z)}.$$

Therefore Proposition 2.10 yields our first main result.

Theorem 2.1 *Let $\mathcal{A}^T, \mathcal{A}^S$ be two invariant symbolic calculi associated with the K-invariant operators T, S on $\mathcal{H} = L^2_a(D,\mu)$. Then the eigenvalues of the associated link transform $\mathcal{B}^{T,S} = (\mathcal{A}^S)'\mathcal{A}^T$ are given by*

$$\begin{aligned}\widetilde{\mathcal{B}^{T,S}}(\underline{\lambda}) &= \int_D \mathcal{A}^T_{e_{\underline{\lambda}}}(K_o)(z)\,\overline{\mathcal{A}^S_{e_{\underline{\lambda}^*}}(K_o)(z)}\; d\mu(z) \qquad (2.65)\\ &= \langle \mathcal{A}^T_{e_{\underline{\lambda}}}(K_o), \mathcal{A}^S_{e_{\underline{\lambda}^*}}(K_o)\rangle_{L^2(D,\mu)}.\end{aligned}$$

For weighted Bergman spaces over a symmetric tube domain $T(\Omega)$ we obtain in particular

Theorem 2.2 *The eigenvalues of the link transform $\mathcal{B}$ associated with the invariant symbolic calculus $\mathcal{A}$ on $L^2_a(T(\Omega),\mu_\nu)$ are given by*

$$\tilde{\mathcal{B}}(\underline{\lambda}) = \langle \mathcal{A}_{e_{\underline{\lambda}}}(K_{ie}), \mathcal{A}_{e_{-\overline{\underline{\lambda}}}}(K_{ie})\rangle_{L^2_a(T(\Omega),\mu_\nu)}.$$

3 The Wick calculus, the fundamental function, and eigenvalues of link transforms

3.1 Sesqui-holomorphic extension of real analytic functions and the Wick calculus

Let D, G, $\mathcal{H}$, $K(z,w)$, U, μ, μ_0 etc. be as in the previous sections. Recall that a *sesqui-holomorphic* function $f(z,w)$ on $D\times D$ is a function which is holomorphic in z and anti-holomorphic in w. The following result is well-known.

Lemma 3.1 *Every sesqui-holomorphic function $f(z,w)$ on $D\times D$ is determined by its restriction to the "diagonal" $f(z,z)$. Namely, if $f(z,z)=0$ for all $z\in D$, then $f(z,w)=0$ for all $z,w\in D$ as well.*

We proceed with the definition of the *Wick calculus* $\mathcal{E}$. We denote by $\mathrm{Dom}(\mathcal{E})$ the space of all real analytic functions $\varphi: D\to\mathbb{C}$ for which there exists a (unique) sesqui-holomorphic function E_φ on all of $D\times D$ satisfying

$$E_\varphi(z,z) = \varphi(z), \quad \forall z\in D.$$

We call the map $\varphi \mapsto E_\varphi$ (defined on $\mathrm{Dom}(\mathcal{E})$) the *extension operator*. Note that $\mathrm{Dom}(\mathcal{E})$ is a linear space, and it contains all real analytic functions on D of the form $f(z) = f_1(z)\,\overline{f_2(z)}$ where f_1 and f_2 are holomorphic in D (since then $E_f(z,w) = f_1(z)\,\overline{f_2(w)}$). Moreover, $\mathrm{Dom}(\mathcal{E})$ contains all exponential functions $e_{\underline{\lambda}}$, $\underline{\lambda} \in C^r$. This can be obtained via case by case considerations (using (1.23) in the case of the Fock space or (1.28) in the case of the weighted Bergman spaces over symmetric tube domains), or via the general formula (1.4)) and the relationship between the complex structures of G and D.

We define a map $\mathcal{E} : \mathrm{Dom}(\mathcal{E}) \to Op(\mathrm{span}\{K_w; w \in D\}) \subset Op(\mathcal{H})$ via its action on the kernel functions

$$\mathcal{E}_\varphi(K_w)(z) := E_\varphi(z,w)\,K(z,w), \quad \forall z, w \in D.$$

Lemma 3.2 *$\mathcal{E}$ is an invariant symbolic calculus, i.e. $Dom(\mathcal{E})$ is G-invariant and*

$$U(g)\,\mathcal{E}_\varphi\,U(g)^{-1} = \mathcal{E}_{\varphi\circ g^{-1}}, \qquad \forall \varphi \in Dom(\mathcal{E}),\ \forall g \in G.$$

Proof: Since G consists of biholomorphic automorphisms of D, it is clear that whenever $\varphi \in \mathrm{Dom}(\mathcal{E})$ and $g \in G$ the function $(z,w) \mapsto E_\varphi(g(z), g(w))$ is sesqui-holomorphic in all of $D \times D$. Moreover, $E_\varphi(g(z), g(z)) = \varphi(g(z))$ for all $z \in D$. Thus, $\varphi \circ g \in \mathrm{Dom}(\mathcal{E})$ and

$$E_{\varphi\circ g}(z,w) = E_\varphi(g(z), g(w)), \ \ \forall z, w \in D.$$

Thus $\mathrm{Dom}(\mathcal{E})$ is G-invariant and E is G-covariant (cf. (2.34)). From this it is clear by the discussion following (2.34) that $\mathcal{E}$ is the invariant symbolic calculus associated with E. Q.E.D

Definition 3.1 *The invariant symbolic calculus $\mathcal{E}$ is called the* Wick calculus.

The name "Wick calculus" is justified by the following result.

Lemma 3.3 *Let $\varphi \in Dom(\mathcal{E})$ admit a factorization $\varphi(z) = \varphi_1(z)\,\overline{\varphi_2(z)}$, $z \in D$, where $\varphi_1, \varphi_2 \in \mathcal{H}$. Then*

$$\mathcal{E}_\varphi = \mathcal{T}_{\varphi_1}\,\mathcal{T}_{\overline{\varphi_2}}, \tag{3.66}$$

where $\mathcal{T}$ is the Toeplitz calculus.

Proof: $\mathcal{T}_{\varphi_1}$, $\mathcal{T}_{\overline{\varphi_2}}$ are well-defined operators, at least on $\mathrm{span}\{K_w; w \in D\}$, and it is well known and easy to prove that

$$\mathcal{T}_{\overline{\varphi_2}}(K_w) = \overline{\varphi_2(w)}\,K_w, \quad \forall w \in D,$$

whereas $\mathcal{T}_{\varphi_1}$ is the operator of multiplication by φ_1. Therefore we obtain for all $z, w \in D$

$$\begin{aligned}(\mathcal{T}_{\varphi_1}\,\mathcal{T}_{\overline{\varphi_2}})(K_w)(z) &= \varphi_1(z)\,\mathcal{T}_{\overline{\varphi_2}}(K_w)(z) = \varphi_1(z)\,\overline{\varphi_2(w)}\,K(z,w) \\ &= E_\varphi(z,w)\,K(z,w) = \mathcal{E}_\varphi(K_w)(z),\end{aligned}$$

and this yields (3.66). Q.E.D

Remark: It follows from (3.66) that if $\varphi \in \mathrm{Dom}(\mathcal{E})$ admits a representation $\varphi = \sum_{j=1}^n f_j \overline{h_j}$ with $f_j, h_j \in \mathcal{H}$, then $\mathcal{E}_\varphi = \sum_{j=1}^n \mathcal{T}_{f_j}\,\mathcal{T}_{\overline{h_j}}$, regardless of the representation of φ. This does not look obvious at first glance.

The following Lemma is a key result. It indicates the central role played by the Wick calculus in our theory.

Lemma 3.4 *Let $\mathcal{A}$ be an invariant symbolic calculus. Define*

$$\mathfrak{a}_{\mathcal{A}}(\underline{\lambda}) = \mathcal{A}_{e_{\underline{\lambda}}}(K_o)(o), \quad \forall e_{\underline{\lambda}} \in Dom(\mathcal{A}).$$

Then

$$\mathcal{A}_{e_{\underline{\lambda}}}(K_w)(z) = \mathfrak{a}_{\mathcal{A}}(\underline{\lambda})\, \mathcal{E}_{e_{\underline{\lambda}}}(K_w)(z) = \mathfrak{a}_{\mathcal{A}}(\underline{\lambda})\, E_{e_{\underline{\lambda}}}(z,w)\, K(z,w) \tag{3.67}$$

for all $e_{\underline{\lambda}} \in Dom(\mathcal{A})$ and $z, w \in D$. Thus the Berezin symbol $A_{e_{\underline{\lambda}}}$ of $\mathcal{A}_{e_{\underline{\lambda}}}$ is given by

$$A_{e_{\underline{\lambda}}}(z,w) = \mathfrak{a}_{\mathcal{A}}(\underline{\lambda})\, E_{e_{\underline{\lambda}}}(z,w), \quad z, w \in D.$$

Proof: Both sides of (3.67) are sesqui-holomorphic in (z,w). In view of Lemma 3.1 it suffices to prove (3.67) for $z = w$. Let $e_{\underline{\lambda}} \in \mathrm{Dom}(\mathcal{A})$ and $z \in D$. Let $g_z \in NA \subset G$ be the unique element for which $g_z(o) = z$. Using (1.18), (2.32), (1.5) and (1.19) we obtain

$$\begin{aligned}\mathcal{A}_{e_{\underline{\lambda}}}(K_z)(z) &= \frac{U(g_z^{-1})(\mathcal{A}_{e_{\underline{\lambda}}}(U(g_z)K_o))(o)}{j(g_z,o)\,\overline{j(g_z,o)}}\\ &= \frac{\mathcal{A}_{e_{\underline{\lambda}}\circ g_z}(K_o)(o)}{|j(g_z,o)|^2} = \mathcal{A}_{e_{\underline{\lambda}}}(K_o)(o)\, e_{\underline{\lambda}}(z)\, K(z,z).\end{aligned}$$

This completes the proof. Q.E.D

Corollary 3.1 *The map $\mathcal{A} \mapsto \mathfrak{a}_{\mathcal{A}}$ is injective. Thus $\mathfrak{a}_{\mathcal{A}}$ completely determines $\mathcal{A}$.*

Proof: Since $\mathrm{span}\{K_w; w \in D\}$ is dense in $\mathcal{H}$ it suffices to show that the function $\mathfrak{a}_{\mathcal{A}}$ determines the action of $\mathcal{A}_b$ on the kernel functions for all $b \in \mathrm{Dom}(\mathcal{A})$. To this end we claim first that for every $b \in \mathcal{X}_{\underline{\lambda}} \cap \mathrm{Dom}(\mathcal{A})$ (see (1.6))

$$\mathcal{A}_b(K_w)(z) = \mathfrak{a}_{\mathcal{A}}(\underline{\lambda})\, \mathcal{E}_b(K_w)(z) = \mathfrak{a}_{\mathcal{A}}(\underline{\lambda})\, E_b(z,w)\, K(z,w), \quad \forall z, w \in D. \tag{3.68}$$

Indeed, it is enough to prove this for $b = e_{\underline{\lambda}} \circ g$ for some $g \in G$, and in this case (1.19) and (3.67) yield

$$\begin{aligned}\mathcal{A}_b(K_w)(z) &= \left(U(g^{-1})\, \mathcal{A}_{e_{\underline{\lambda}}}\, U(g)\right)(K_w)(z)\\ &= j(g,z)\,\overline{j(g,w)}\, \mathcal{A}_{e_{\underline{\lambda}}}(K_{g(w)})(g(z))\\ &= \mathfrak{a}_{\mathcal{A}}(\underline{\lambda})\, E_{e_{\underline{\lambda}}}(g(z),g(w))\, j(g,z)\,\overline{j(g,w)}\, K(g(z),g(w))\\ &= \mathfrak{a}_{\mathcal{A}}(\underline{\lambda})\, E_{e_{\underline{\lambda}}\circ g}(z,w)\, K(z,w).\end{aligned}$$

Since $\mathrm{span}\{\cup_{\underline{\lambda}\in\mathbb{C}^r} \mathcal{X}_{\underline{\lambda}} \cap \mathrm{Dom}(\mathcal{A})\}$ is dense in $\mathrm{Dom}(\mathcal{A})$, (3.68) shows that $\mathfrak{a}_{\mathcal{A}}$ determines $\mathcal{A}$. Q.E.D

Definition 3.2 *The function $\mathfrak{a}_{\mathcal{A}}$ is called the* fundamental function *associated with the calculus $\mathcal{A}$.*

As will become clear below, the eigenvalues of the link transforms can be expressed conveniently via the fundamental functions of the associated calculi. Notice also that we clearly have

$$\mathfrak{a}_{\mathcal{E}}(\underline{\lambda}) = 1, \quad \forall \underline{\lambda} \in \mathbb{C}^r.$$

This follows easily from (3.67).

Let B be the K-invariant operator on $\mathcal{H}$ which determines $\mathcal{A}$ via Corollary 2.3, i.e. $\mathcal{A} = \mathcal{A}^B$. For simplicity we write also $\mathfrak{a}_{\mathcal{A}^B}(\underline{\lambda}) = \mathfrak{a}_B(\underline{\lambda})$. Let $\{B_\xi\}_{\xi\in D}$ be the covariant field of operators generated by B via (2.42). Let us define

$$\mathfrak{b}_{\mathcal{A}}(\xi) := B_\xi(K_o)(o) = \langle B_\xi(K_o), K_o\rangle_{\mathcal{H}}, \quad \xi \in D.$$

Lemma 3.5 (i) *The function $\mathfrak{b}_{\mathcal{A}}(\xi)$ is K-invariant in the sense that $\mathfrak{b}_{\mathcal{A}}(k(\xi)) = \mathfrak{b}_{\mathcal{A}}(\xi)$ for all $k \in K$ and $\xi \in D$.*
(ii) *$\mathfrak{a}_{\mathcal{A}}$ is the spherical Fourier transform of $\mathfrak{b}_{\mathcal{A}}$, i.e., for $e_{\underline{\lambda}} \in Dom(\mathcal{A})$ we have*

$$\mathfrak{a}_{\mathcal{A}}(\underline{\lambda}) = \widetilde{\mathfrak{b}_{\mathcal{A}}}(\underline{\lambda}) = \int_D e_{\underline{\lambda}}(\xi)\, \mathfrak{b}_{\mathcal{A}}(\xi)\, d\mu_0(\xi). \tag{3.69}$$

(iii) *$\mathfrak{a}_{\mathcal{A}}$ is invariant under the Weyl group W:*

$$\mathfrak{a}_{\mathcal{A}}(w(\underline{\lambda})) = \mathfrak{a}_{\mathcal{A}}(\underline{\lambda}), \quad \forall w \in W. \tag{3.70}$$

(iv) *Let $\psi(\xi) = g_\xi^{-1}(o)$, see* (2.55) and Proposition 2.8. Then

$$\mathfrak{b}_{\mathcal{A}}(\xi) = B(K_{\psi(\xi)})(\psi(\xi))\, K(\psi(\xi), \psi(\xi)), \quad \forall \xi \in D.$$

Proof: (i) Let $\xi \in D$ and $k \in K$. Then $B_{k(\xi)} = U(k)\, B_\xi\, U(k)^{-1}$. Hence, (2.42) yields

$$\begin{aligned} \mathfrak{b}_{\mathcal{A}}(k(\xi)) &= \langle B_\xi U(k)^{-1}(K_o), U(k)^{-1}(K_o)\rangle_{\mathcal{H}} \\ &= \langle B_\xi(\overline{j(k^{-1},o)}K_o), \overline{j(k^{-1},o)}K_o\rangle_{\mathcal{H}} = \langle B_\xi(K_o), K_o\rangle_{\mathcal{H}} = \mathfrak{b}_{\mathcal{A}}(\xi), \end{aligned}$$

since $j(k^{-1}, o)$ is unimodular and $k^{-1}(o) = o$.
(ii) This follows from the definition of $\mathfrak{a}_{\mathcal{A}}$:

$$\mathfrak{a}_{\mathcal{A}}(\underline{\lambda}) = \mathcal{A}_{e_{\underline{\lambda}}}(K_o)(o) = \int_D e_{\underline{\lambda}}(\xi)\, B_\xi(K_o)(o)\, d\mu_0(\xi) = \int_D e_{\underline{\lambda}}(\xi)\, \mathfrak{b}_{\mathcal{A}}(\xi)\, d\mu_0(\xi).$$

(iii) This follows by (i), (ii) and (1.8). Indeed, let $w \in W$, then

$$\mathfrak{a}_{\mathcal{A}}(\underline{\lambda}) = \int_D \phi_{\underline{\lambda}}(\xi)\, \mathfrak{b}_{\mathcal{A}}(\xi)\, d\mu_0(\xi) = \int_D \phi_{w(\underline{\lambda})}(\xi)\, \mathfrak{b}_{\mathcal{A}}(\xi)\, d\mu_0(\xi) = \mathfrak{a}_{\mathcal{A}}(w(\underline{\lambda})).$$

(iv) Using (2.42) we obtain for all $\xi \in D$

$$\begin{aligned} \mathfrak{b}_{\mathcal{A}}(\xi) &= B_\xi(K_o)(o) = (U(g_\xi)\, B\, U(g_\xi^{-1}))(K_o)(o) \\ &= j(g_\xi^{-1}, o)\; B(\overline{j(g_\xi, o)}\, K_{\psi(\xi)})(\psi(\xi)) = B(K_{\psi(\xi)})(\psi(\xi))\, K(\psi(\xi), \psi(\xi)). \end{aligned}$$

Q.E.D.

The proof of Corollary 3.1 yields the following result.

Proposition 3.1 *Let $\mathfrak{a}$ be a W-invariant holomorphic function, defined on a W-invariant subset of $\mathbb{C}^r$. Then there exists a unique invariant symbolic calculus $\mathcal{A}$ on $\mathcal{H}$ for which $\mathfrak{a} = \mathfrak{a}_{\mathcal{A}}$.*

Proof: Notice first that since $e_{\underline{\lambda}} \in \mathrm{Dom}(\mathcal{E})$ for all $\underline{\lambda}$ we have $\mathcal{X}_{\underline{\lambda}} \subset \mathrm{Dom}(\mathcal{E})$ (cf. (1.6)) by the G-invariance of $\mathrm{Dom}(\mathcal{E})$. For $\underline{\lambda} \in \mathrm{Dom}(\mathfrak{a})$ and $b \in \mathcal{X}_{\underline{\lambda}}$ we define $\mathcal{A}_b$ on the kernel functions via (3.68), and extend it to $\mathrm{span}\{K_w; w \in D\}$ by linearity. To check the invariance (2.32) let $g \in G$. Then, for all $z, w \in D$,

$$\begin{aligned}(U(g)\,\mathcal{A}_b\,U(g^{-1}))(K_w)(z) &= j(g^{-1},z)\,\overline{j(g^{-1},w)}\,\mathcal{A}_b(K_{g^{-1}(w)})(g^{-1}(z))\\ &= j(g^{-1},z)\,\overline{j(g^{-1},w)}\,\mathfrak{a}(\underline{\lambda})\,E_b(g^{-1}(z),g^{-1}(w))\,K(g^{-1}(z),g^{-1}(w))\\ &= \mathfrak{a}(\underline{\lambda})\,E_{b\circ g^{-1}}(z,w)\,K(z,w) = \mathcal{A}_{b\circ g^{-1}}(K_w)(z).\end{aligned}$$

We extend $\mathcal{A}$ to $\mathrm{span}\{\cup_{\underline{\lambda}\in \mathrm{Dom}(\mathfrak{a})}\mathcal{X}_{\underline{\lambda}}\}$ by linearity, and define $\mathrm{Dom}(\mathcal{A})$ to be this space. Then $\mathrm{Dom}(\mathcal{A})$ is G-invariant and $\mathcal{A}$ is an invariant symbolic calculus. Finally, the uniqueness of $\mathcal{A}$ follows from (3.68) and the fact that $\mathcal{A}_b$ is determined by its action on the kernel functions $K_w, w \in D$. Q.E.D.

The function $\mathfrak{b}_{\mathcal{A}}(\xi)$ is real analytic and K-invariant on D. The next result shows that such functions are in one-to-one correspondence with the invariant symbolic calculi.

Proposition 3.2 *Let $\mathfrak{b}$ be a K-invariant real analytic function on D. Then there exists a unique covariant field of operators $\{B_\xi\}_{\xi\in D}$ on $\mathcal{H}$ so that $\mathfrak{b}(\xi) = B_\xi(K_o)(o)$ for all $\xi \in D$. Consequently, $\mathfrak{b} = \mathfrak{b}_{\mathcal{A}}$ where $\mathcal{A}$ is the invariant symbolic calculus generated by $\{B_\xi\}_{\xi\in D}$ via* (2.39).

Proof: Fix $\xi \in D$ and define the operator B_ξ in the following steps. First, we define for every $w \in D$

$$B_\xi(K_w)(w) = K(w,w)\,\mathfrak{b}(g^{-1}(\xi)), \quad \text{where } g \in G, g(o) = w.$$

The K-invariance of $\mathfrak{b}$ implies that the definition is independent of the particular g. The real analyticity of $\mathfrak{b}$ implies the real analyticity of the function $w \mapsto B_\xi(K_w)(w)$. We assume the existence of a unique sesqui-holomorphic extension to $D \times D$, which is denoted by $B_\xi(K_w)(z)$. Now extend this operator by linearity to $\mathrm{span}\{K_w; w \in D\}$. The family of operators $\{B_\xi\}_{\xi\in D}$ constructed in this way is covariant. Indeed, let $\xi, w \in D$ and let $h \in G$ so that $h(o) = w$. Then for any $g \in G$,

$$\begin{aligned}(U(g^{-1})\,B_\xi\,U(g))(K_w)(w) &= |j(g,w)|^2\,B_\xi(K_{g(w)})(g(w))\\ &= |j(g,w)|^2\,K(g(w),g(w))\,b((g\circ h)^{-1}(\xi))\\ &= K(w,w)\,b(h^{-1}(g^{-1}(\xi))) = B_{g^{-1}}(K_w)(w).\end{aligned}$$

This shows that $U(g^{-1})\,B_\xi\,U(g) = B_{g^{-1}(\xi)}$. Finally, the relationship $\mathfrak{b}(\xi) = B_\xi(K_o)(o)$ follows from the definition of B_ξ. Q.E.D.

Proposition 3.3 *Let $\mathcal{A}$ be an invariant symbolic calculus on $\mathcal{H}$ and let $\mathcal{B}$ be a G-invariant operator on D. Define the composition $\mathcal{A}\mathcal{B}$ via*

$$\begin{aligned}Dom(\mathcal{A}\mathcal{B}) &= \{b \in Dom(\mathcal{B});\ \mathcal{B}_b \in Dom(\mathcal{A})\} \text{ and}\\ (\mathcal{A}\mathcal{B})_b &:= \mathcal{A}_{\mathcal{B}(b)},\ \forall b \in Dom(\mathcal{A}\mathcal{B}).\end{aligned}$$

Then, (i) $\mathcal{A}\,\mathcal{B}$ *is an invariant symbolic calculus, i.e. it satisfies* (2.32).

(ii) The fundamental function of $\mathcal{A}\,\mathcal{B}$ is

$$\mathfrak{a}_{\mathcal{A}\,\mathcal{B}}(\underline{\lambda}) = \mathfrak{a}_{\mathcal{A}}(\underline{\lambda})\, \widetilde{\mathcal{B}}(\underline{\lambda}).$$

(iii) Let $\mathcal{A} = \mathcal{U}\,|\mathcal{A}|$ be the polar decomposition of $\mathcal{A}$ with respect to inner products of $L^2(D,\mu_0)$ and $S_2(\mathcal{H})$ respectively, and let $\mathcal{B} = \mathcal{V}\,|\mathcal{B}|$ be the polar decomposition of $\mathcal{B}$ with respect to $L^2(D,\mu_0)$. Then $|\mathcal{A}|$, $|\mathcal{B}|$ and $\mathcal{V}$ are G-invariant operators on D, $\mathcal{U}$ is an invariant symbolic calculus on $\mathcal{H}$, and the polar decomposition of $\mathcal{A}\,\mathcal{B}$ is

$$\mathcal{A}\,\mathcal{B} = (\mathcal{U}\,\mathcal{V})\,(|\mathcal{A}|\,|\mathcal{B}|),$$

namely $|\mathcal{A}\,\mathcal{B}| = |\mathcal{A}|\,|\mathcal{B}|$ and the partial isometry in the minimal polar decomposition of $\mathcal{A}\,\mathcal{B}$ is $\mathcal{U}\,\mathcal{V}$.

Proof: (i) Since $\mathrm{Dom}(\mathcal{B})$ is G-invariant, $\mathrm{Dom}(\mathcal{B}) \cap \mathcal{X}_{\underline{\lambda}} \neq \{0\}$ implies that $\mathcal{X}_{\underline{\lambda}} \subseteq \mathrm{Dom}(\mathcal{B})$ and $\mathcal{B}_{|\mathcal{X}_{\underline{\lambda}}} = \widetilde{\mathcal{B}}(\underline{\lambda})\, I_{|\mathcal{X}_{\underline{\lambda}}}$. Thus $\mathcal{B} : \mathrm{Dom}(\mathcal{B}) \to \mathrm{Dom}(\mathcal{B})$ and the composition $\mathcal{A}\,\mathcal{B}$ is defined in $\mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(\mathcal{B})$. Next, for every $g \in G$ and $\mathfrak{b} \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(\mathcal{B})$,

$$U(g)(\mathcal{A}\,\mathcal{B})_{\mathfrak{b}}\, U(g^{-1}) = U(g)\, \mathcal{A}_{\mathcal{B}(\mathfrak{b})}\, U(g^{-1}) = \mathcal{A}_{\mathcal{B}(\mathfrak{b})\circ g^{-1}} = \mathcal{A}_{\mathcal{B}(\mathfrak{b}\circ g^{-1})} = (\mathcal{A}\,\mathcal{B})_{\mathfrak{b}\circ g^{-1}}.$$

(ii) Let $e_{\underline{\lambda}} \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(\mathcal{B})$. Since $\mathcal{B}(e_{\underline{\lambda}}) = \widetilde{\mathcal{B}}(\underline{\lambda})\, e_{\underline{\lambda}}$, we obtain

$$\mathfrak{a}_{\mathcal{A}\,\mathcal{B}}(\underline{\lambda}) = (\mathcal{A}\,\mathcal{B})_{e_{\underline{\lambda}}}(K_o)(o) = \mathcal{A}_{\mathcal{B}(e_{\underline{\lambda}})}(K_o)(o) = \widetilde{\mathcal{B}}(\underline{\lambda})\, \mathcal{A}_{e_{\underline{\lambda}}}(K_o)(o) = \widetilde{\mathcal{B}}(\underline{\lambda})\, \mathfrak{a}_{\mathcal{A}}(\underline{\lambda}).$$

(iii) For the modulus of $\mathcal{A}\,\mathcal{B}$ we have $|\mathcal{A}\,\mathcal{B}|^2 = (\mathcal{A}\,\mathcal{B})'(\mathcal{A}\,\mathcal{B}) = \mathcal{B}'\,|\mathcal{A}|^2\,\mathcal{B} = |\mathcal{A}|^2\,|\mathcal{B}|^2$, since all invariant operators commute. Taking the square root and using the commutativity again, we obtain $|\mathcal{A}\,\mathcal{B}| = |\mathcal{A}|\,|\mathcal{B}|$ and

$$\mathcal{A}\,\mathcal{B} = \mathcal{U}\,|\mathcal{A}|\,\mathcal{V}\,|\mathcal{B}| = \mathcal{U}\,\mathcal{V}\,|\mathcal{A}|\,|\mathcal{B}| = \mathcal{U}\,\mathcal{V}\,|\mathcal{A}\,\mathcal{B}|.$$

Since $\mathcal{V}$ maps $\mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(\mathcal{B})$ isometrically onto itself and this space is the domain of definition of $\mathcal{U}$, we see that $\mathcal{U}\,\mathcal{V}$ is a partial isometry, whose kernel is the same as that of $|\mathcal{A}\,\mathcal{B}|$. This completes the proof. Q.E.D.

Every invariant symbolic calculus $\mathcal{A}$ on $\mathcal{H}$ can be factorized in many ways as $\mathcal{A} = \mathcal{A}_1\,\mathcal{B}$ with $\mathcal{A}_1$ invariant symbolic calculus and $\mathcal{B}$ a G-invariant operator on D. Besides the trivial factorization $\mathcal{A} = \mathcal{A}I$ and the factorization coming from the polar decomposition $\mathcal{A} = \mathcal{U}\,|\mathcal{A}|$, there is a canonical factorization in which $\mathcal{A}_1$ is the Wick calculus $\mathcal{E}$.

Proposition 3.4 *Let* $\mathcal{A}$ *be an invariant symbolic calculus on* $\mathcal{H}$. *Then*

$$\mathcal{A} = \mathcal{E}\; C_{\mathcal{A}}, \tag{3.71}$$

where $C_{\mathcal{A}}$ *is the operator of convolution with* $\mathfrak{b}_{\mathcal{A}}$*:*

$$(C_{\mathcal{A}}f)(z) = \int_D f(g(\xi))\, \mathfrak{b}_{\mathcal{A}}(\xi)\, d\mu_0(\xi), \quad \textit{where } g \in G,\ g(o) = z.$$

Proof: $C_{\mathcal{A}}$ is certainly G-invariant, hence Proposition 3.3 guarantees that $\mathcal{E}\, C_{\mathcal{A}}$ is an invariant symbolic calculus. Moreover, the eigenvalues of $C_{\mathcal{A}}$ are $\widetilde{C_{\mathcal{A}}}(\underline{\lambda}) = \widetilde{\mathfrak{b}_{\mathcal{A}}}(\underline{\lambda}) = \mathfrak{a}_{\mathcal{A}}(\underline{\lambda})$. Hence,

$$\mathfrak{a}_{\mathcal{E}\, C_{\mathcal{A}}}(\underline{\lambda}) = \mathfrak{a}_{\mathcal{E}}(\underline{\lambda})\, \widetilde{C_{\mathcal{A}}}(\underline{\lambda}) = \mathfrak{a}_{\mathcal{A}}(\underline{\lambda}).$$

since $\mathfrak{a}_{\mathcal{E}}(\underline{\lambda}) \equiv 1$. Thus Corollary 3.1 guarantees that (3.71) holds. Q.E.D.

Remark: Propositions 3.2 and 3.4 can be generalized to distributions. We outline this generalization briefly. Let $\mathfrak{b}$ be a K-invariant distribution on D, and let $C_{\mathfrak{b}}$ be the operator of convolution with $\mathfrak{b}$:

$$(C_{\mathfrak{b}} f)(z) := \langle f \circ g, \mathfrak{b} \rangle, \quad \text{where } g \in G,\ g(o) = z.$$

Then $\mathcal{A} := \mathcal{E}\, C_{\mathfrak{b}}$ is an invariant symbolic calculus on $\mathcal{H}$ in its canonical factorization. In the important case of the Dirac measure $\mathfrak{b} = \delta_o$ we obtain $\mathcal{A} = \mathcal{E}$ since $C_{\delta_o} = I$. However, this general approach leads to some open problems which will be discussed in a subsequent publication.

3.3 Eigenvalues of link transforms via the fundamental functions

Proposition 3.5 *Let T be a K-invariant operator on $\mathcal{H} = L^2_a(D,\mu)$ and let $\mathfrak{a}_T$ be the fundamental function associated with the calculus $\mathcal{A}^T$. Let T^* the adjoint of T as an operator on $\mathcal{H}$. Then for all admissible $\underline{\lambda} \in \mathbb{C}^r$ we have*

$$\mathfrak{a}_{T^*}(\underline{\lambda}) = \overline{\mathfrak{a}_T(\overline{\underline{\lambda}})}. \tag{3.72}$$

Proof: Let $\mathfrak{b}_T(\xi) = \mathfrak{b}_{\mathcal{A}^T}(\xi) = \langle T_\xi K_o, K_o \rangle_{\mathcal{H}}$ and define $\mathfrak{b}_{T^*}(\xi)$ similarly. Then, $\mathfrak{b}_{T^*}(\xi) = \overline{\mathfrak{b}_T(\xi)}$. Using the fact that $\overline{e_{\underline{\lambda}}(\xi)} = e_{\overline{\underline{\lambda}}}(\xi)$ we obtain by (3.69)

$$\begin{aligned}
\mathfrak{a}_{T^*}(\underline{\lambda}) &= \int_D e_{\underline{\lambda}}(\xi)\, \mathfrak{b}_{T^*}(\xi)\, d\mu_0(\xi) \\
&= \int_D e_{\underline{\lambda}}(\xi)\, \overline{\mathfrak{b}_T(\xi)}\, d\mu_0(\xi) = \overline{\int_D e_{\overline{\underline{\lambda}}}(\xi)\, \mathfrak{b}_T(\xi)\, d\mu_0(\xi)} = \overline{\mathfrak{a}_T(\overline{\underline{\lambda}})}.
\end{aligned}$$

Q.E.D

Remark: In the case where $\mathcal{H}$ is either the weighted Bergman space over a symmetric tube domain $L^2_a(T(\Omega),\mu_\nu)$ or the weighted Fock space $\mathcal{F}_\nu$ we have also

$$\mathfrak{a}_{T^*}(\underline{\lambda}) = \overline{\mathfrak{a}_T(\underline{\lambda}^*)}. \tag{3.73}$$

Indeed, in the case of $L^2_a(T(\Omega),\mu_\nu)$ we have $\underline{\lambda}^* = -\overline{\underline{\lambda}}$ (see Lemma 2.3 and (2.59)), and (3.73) follows from (3.70) and (3.72). In the case of $\mathcal{F}_\nu$, if $a, b \in \mathbb{C}^d$ are so that $\langle a, b\rangle = \lambda$ then $e_{\lambda^*} = e_{-b,-a}$ (see (1.25) and (2.57)). Hence

$$\overline{\mathfrak{a}_T(\lambda^*)} = \overline{\int_{\mathbb{C}^d} e_{-b,-a}(\xi)\, \mathfrak{b}_T(\xi)\, d\mu_0(\xi)} = \int_{\mathbb{C}^d} e_{b,a}(-\xi)\, \overline{\mathfrak{b}T(\xi)}\, d\mu_0(\xi) = \mathfrak{a}_{T^*}(\lambda),$$

since $\mathfrak{a}_T(\xi) = \mathfrak{a}_T(-\xi)$ by Lemma 3.5.

We now combine the results of subsections 2.3 and 3.2 to obtain our main result.

Theorem 3.1 *Let S, T be K-invariant operators on $\mathcal{H} = L^2_a(D, \mu)$ and let $\mathfrak{a}_S$, $\mathfrak{a}_T$ be the fundamental functions of the associated invariant symbolic calculi $\mathcal{A}^S$ and $\mathcal{A}^T$ respectively. Let $\mathcal{B}^{S,T} = (\mathcal{A}^T)' \mathcal{A}^S$ be the corresponding link transform. Then the eigenvalues of $\mathcal{B}^{S,T}$ are expressed in terms of the fundamental functions in the following way*

$$\widetilde{\mathcal{B}^{S,T}}(\underline{\lambda}) = \frac{\mathfrak{a}_S(\underline{\lambda})\, \overline{\mathfrak{a}_T(\underline{\lambda}^*)}}{\mathfrak{a}_{\mathcal{T}}(\underline{\lambda})}, \quad \forall \underline{\lambda} \in Dom(\mathcal{B}^{S,T}), \tag{3.74}$$

where $\mathfrak{a}_{\mathcal{T}}(\underline{\lambda})$ is the fundamental function associated with the Toeplitz calculus $\mathcal{T}$ and $\underline{\lambda} \mapsto \underline{\lambda}^*$ is the involution whose existence is guaranteed by Lemma 2.3.

Proof: We know by Theorem 2.1 that

$$\widetilde{\mathcal{B}^{S,T}}(\underline{\lambda}) = \int_D \mathcal{A}^S_{e_{\underline{\lambda}}}(K_o)(z)\, \overline{\mathcal{A}^T_{\underline{\lambda}^*}(K_o)(z)}\, d\mu(z).$$

However, Lemma 3.4 says that

$$\mathcal{A}^S_{e_{\underline{\lambda}}}(K_o)(z) = \mathfrak{a}_S(\underline{\lambda})\, E_{e_{\underline{\lambda}}}(z,o)\, K(z,o), \quad \mathcal{A}^T_{e_{\underline{\lambda}^*}}(K_o)(z) = \mathfrak{a}_T(\underline{\lambda}^*)\, E_{e_{\underline{\lambda}^*}}(z,o)\, K(z,o).$$

Thus,

$$\widetilde{\mathcal{B}^{S,T}}(\underline{\lambda}) = \mathfrak{a}_S(\underline{\lambda})\, \overline{\mathfrak{a}_T(\underline{\lambda}^*)}\, c(\underline{\lambda}), \tag{3.75}$$

where

$$c(\underline{\lambda}) = \int_D E_{e_{\underline{\lambda}}}(z,o)\, \overline{E_{e_{\underline{\lambda}^*}}(z,o)}\, |K(z,o)|^2\, d\mu(z)$$

is independent of the particular calculi $\mathcal{A}^S$, $\mathcal{A}^T$. Let us apply (3.75) with $\mathcal{A}^S = \mathcal{A}^T = \mathcal{T}$. This yields

$$c(\underline{\lambda}) = \frac{\widetilde{\mathcal{T}'\mathcal{T}}(\underline{\lambda})}{\mathfrak{a}_{\mathcal{T}}(\underline{\lambda})\, \overline{\mathfrak{a}_{\mathcal{T}}(\underline{\lambda}^*)}}.$$

In the case of the Toeplitz calculus we have $\mathcal{T} = \mathcal{A}^{P_o}$, where $P_o := \langle \cdot, K_o \rangle_{\mathcal{H}} K_o$, the projection on $\mathbb{C}K_o$, is a self adjoint operator. Hence $\mathfrak{a}_{\mathcal{T}}(\underline{\lambda}^*) = \mathfrak{a}_{\mathcal{T}}(\underline{\lambda})$, and

$$\widetilde{\mathcal{T}'\mathcal{T}}(\underline{\lambda}) = \langle \mathcal{T}_{e_{\underline{\lambda}}}, P_o \rangle_{S_2(\mathcal{H})} = \mathcal{T}_{e_{\underline{\lambda}}}(K_o)(o) = \mathfrak{a}_{\mathcal{T}}(\underline{\lambda}).$$

Therefore

$$c(\underline{\lambda}) = \frac{1}{\mathfrak{a}_{\mathcal{T}}(\underline{\lambda})}.$$

Using this fact, Proposition 3.5 and (3.75) we obtain (3.74). Q.E.D.

Remark: In the case where $\mathcal{H}$ is either the weighted Bergman space over a symmetric tube domain $L^2_a(T(\Omega), \mu_\nu)$ or the weighted Fock space $\mathcal{F}_\nu$ we can write (3.74) also as

$$\widetilde{\mathcal{B}^{S,T}}(\underline{\lambda}) = \frac{\mathfrak{a}_S(\underline{\lambda})\, \overline{\mathfrak{a}_T(\overline{\underline{\lambda}})}}{\mathfrak{a}_{\mathcal{T}}(\underline{\lambda})}, \quad \forall \underline{\lambda} \in \mathrm{Dom}(\mathcal{B}^{S,T}),$$

Indeed, this follows from (3.74) and (3.73).

Theorem 3.1 is a substantial improvement of Theorem 2.1, since instead of the inner product of the functions $\mathcal{A}^T_{e_{\underline{\lambda}}}(K_o)(z)$, $\mathcal{A}^S_{e_{\underline{\lambda}^*}}(K_o)(z)$ one needs only to compute the product $\mathfrak{a}_T(\underline{\lambda})\, \overline{\mathfrak{a}_{S^*}(\underline{\lambda})} =$

$\mathcal{A}^{*}_{e_{\underline{\lambda}}}(K_o)(o)\,\mathcal{A}^{\circ}_{e_{\underline{\lambda}^*}}(K_o)(o)$. Theorem 3.1 involves the fundamental function of the Toeplitz calculus $\mathfrak{a}_{\mathcal{T}}(\underline{\lambda})$, which is known in the cases under consideration. Indeed, in the case where $\mathcal{H}$ is the Fock space $\mathcal{F}_\nu$ we have

$$\mathfrak{a}_{\mathcal{T}}(\lambda) = e^{\frac{\lambda}{\nu}},$$

and in the case where $\mathcal{H}$ is the weighted Bergman space $L^2_a(T(\Omega), \mu_\nu)$ we have, by [UU94] (see also [Be78]),

$$\mathfrak{a}_{\mathcal{T}}(\underline{\lambda}) = \frac{\Gamma_\Omega(\underline{\lambda} + \underline{\rho} + \nu - \frac{d}{r})\,\Gamma_\Omega(-\underline{\lambda} + \underline{\rho} + \nu - \frac{d}{r})}{\Gamma_\Omega(\nu - \frac{d}{r})\,\Gamma_\Omega(\nu)} \tag{3.76}$$

We will give a new proof of this result in Section 5.

We close by observing that the results in this section yield also interesting information on the fundamental function of the unitary part of an invariant symbolic calculus $\mathcal{A}$. Let $\mathcal{A} = \mathcal{U}(\mathcal{A}'\mathcal{A})^{\frac{1}{2}}$ be the minimal polar decomposition of $\mathcal{A}$ (thus $\ker(\mathcal{A}) = \ker(\mathcal{U})$). We know that $\mathcal{U}$ itself is also an invariant symbolic calculus. Thus for $e_{\underline{\lambda}} \in \mathrm{Dom}(\mathcal{A})$ we have

$$\mathcal{A}_{e_{\underline{\lambda}}} = \mathcal{U}_{(\mathcal{A}'\mathcal{A})^{\frac{1}{2}} e_{\underline{\lambda}}} = \left(\widetilde{\mathcal{A}'\mathcal{A}}(\underline{\lambda})\right)^{\frac{1}{2}} \mathcal{U}_{e_{\underline{\lambda}}}.$$

From this it follows that

$$\mathfrak{a}_{\mathcal{A}}(\underline{\lambda}) = \left(\widetilde{\mathcal{A}'\mathcal{A}}(\underline{\lambda})\right)^{\frac{1}{2}} \mathfrak{a}_{\mathcal{U}}(\underline{\lambda}) = \left(\frac{\mathfrak{a}_{\mathcal{A}}(\underline{\lambda})\,\overline{\mathfrak{a}_{\mathcal{A}}(\overline{\underline{\lambda}})}}{\mathfrak{a}_{\mathcal{T}}(\underline{\lambda})}\right)^{\frac{1}{2}} \mathfrak{a}_{\mathcal{U}}(\underline{\lambda}).$$

Thus, if $\mathfrak{a}_{\mathcal{A}}(\underline{\lambda}) \neq 0$ then

$$\mathfrak{a}_{\mathcal{U}}(\underline{\lambda}) = \left(\frac{\mathfrak{a}_{\mathcal{A}}(\underline{\lambda})}{\overline{\overline{\mathfrak{a}_{\mathcal{A}}(\overline{\underline{\lambda}})}}}\right)^{\frac{1}{2}} \mathfrak{a}_{\mathcal{T}}(\underline{\lambda})^{\frac{1}{2}} = \left(\frac{\mathfrak{a}_{\mathcal{A}}(\underline{\lambda})}{\mathfrak{a}_{\mathcal{A}^*}(\underline{\lambda})}\right)^{\frac{1}{2}} \mathfrak{a}_{\mathcal{T}}(\underline{\lambda})^{\frac{1}{2}}.$$

In particular, if $T = T^*$ then

$$\mathfrak{a}_{\mathcal{U}}(\underline{\lambda}) = \mathfrak{a}_{\mathcal{T}}(\underline{\lambda})^{\frac{1}{2}}.$$

4 Application to weighted Fock spaces

In this section we apply the general theory developed above to study various important invariant symbolic calculi in the context of the weighted Fock spaces $\mathcal{F}_\nu$ over $\mathbb{C}^d$ introduced in subsection 1.2 above. After describing the standard calculi of Wick, Weyl and anti-Wick (Toeplitz) type in the following subsection, we construct new families of calculi, related to the "k - homogeneous" projections in the next subsection, and give a detailed numerical analysis of their spectral properties. This yields a rather surprising interrelationship between different calculi.

4.1 The Toeplitz, Weyl and Wick calculi

Proposition 2.9 yields the following result.

Theorem 4.1 *Let $\mathcal{A}$ be an invariant symbolic calculus on the weighted Fock space $\mathcal{F}_\nu = L^2_a(\mathbb{C}^d, \mu_\nu)$. Then the eigenvalues of the associated link transform $\mathcal{B} = \mathcal{A}'\mathcal{A}$ are given by*

$$\tilde{\mathcal{B}}(\lambda) = \int_{\mathbb{C}^d} \mathcal{A}_{e_{a,b}}(1)(z)\ \overline{\mathcal{A}_{e_{-b,-a}}(1)(z)}\ d\mu_\nu(z) = \langle \mathcal{A}_{e_{a,b}}(1),\ \mathcal{A}_{e_{-b,-a}}(1)\rangle_{\mathcal{F}_\nu} \tag{4.77}$$

where $a, b \in \mathbb{C}^d$ are arbitrary vectors for which $\lambda = \langle a|b\rangle$.

Proof: With the notation as in the previous section, we have

$$\begin{aligned} {}_z\tilde{F}(\lambda) &= \int_{\mathbb{C}^d} F(z,w)\ e_{a,b}(w)\ d\mu_0(w) = \beta(e_{a,b})(z) \\ &= \mathcal{A}_{e_{a,b}}(K_o)(z)/K_o(z) = \mathcal{A}_{e_{a,b}}(1)(z), \end{aligned}$$

since $K_o(w) \equiv 1$. Similarly,

$$\begin{aligned} {}_z\widetilde{\overline{F} \circ \psi}(\lambda) &= \int_{\mathbb{C}^d} \overline{F(z,\psi(w))}\ e_{a,b}(w)\ d\mu_o(w) = \overline{\int_{\mathbb{C}^d} F(z,\xi)\ \overline{e_{a,b}(\psi(\xi))}\ d\mu_0(\psi(\xi))} \\ &= \overline{\int_{\mathbb{C}^d} F(z,\xi)\ e_{-b,-a}(\xi)\ d\mu_0(\xi)} \ = \ \overline{\mathcal{A}_{e_{-b,-a}}(1)(z)}. \end{aligned}$$

These results imply (4.77) via (2.65). Q.E.D.

The extension of Theorem 4.1 to the link transform $\mathcal{B}^{T,S}$ is a consequence of Proposition 2.10.

Theorem 4.2 *Let $\mathcal{A}^T, \mathcal{A}^S$ be invariant symbolic calculi generated by the K-invariant operators T, S on holomorphic functions on $\mathbb{C}^d$. Then the eigenvalues of the associated link transform $\mathcal{B}^{T,S} = (\mathcal{A}^S)'\mathcal{A}^T$ are*

$$\widetilde{\mathcal{B}^{T,S}}(\lambda) = \int_{\mathbb{C}^d} \mathcal{A}^T_{e_{a,b}}(1)(z)\overline{\mathcal{A}^S_{e_{-b,-a}}(1)(z)}\ d\mu_\nu(z) = \langle \mathcal{A}^T_{e_{a,b}}(1),\ \mathcal{A}^S_{e_{-b,-a}}(1)\rangle_{\mathcal{F}_\nu}$$

where $a, b \in \mathbb{C}^d$ are arbitrary vectors for which $\langle a|b\rangle = \lambda$.

We turn now to some important examples to illustrate the scope of our theory.

Example 4.1: Toeplitz calculus. Let $\mathcal{T}_b = \mathcal{T}^{(\nu)}_b$ be the Toeplitz operator with symbol b on $\mathcal{F}_\nu = L^2_a(\mathbb{C}^d, \mu_\nu)$. Then

$$\mathcal{T}_{e_{a,b}}(1)(z) = e^{\frac{\lambda}{\nu}}\ e^{\langle z|b\rangle},$$

and in particular

$$\mathfrak{a}_{\mathcal{T}}(\lambda) = e^{\frac{\lambda}{\nu}}.$$

Indeed, if $\lambda = \langle a, b\rangle$ then

$$\mathcal{T}_{e_{a,b}}(1)(z) = \int_{\mathbb{C}^d} e^{\langle w|b\rangle}\ e^{\nu\langle \frac{a}{\nu}+z|w\rangle}\ d\mu_\nu(w) = e^{\frac{\lambda}{\nu}}\ e^{\langle z|b\rangle}.$$

Similarly,

$$\mathcal{T}_{e_{-b,-a}}(1)(z) = e^{\frac{\overline{\lambda}}{\nu}}\ e^{-\langle z|a\rangle}\ .$$

Hence

$$\int_{\mathbb{C}^d} \mathcal{T}_{e_{a,b}}(1)(z)\ \overline{\mathcal{T}_{e_{-b,-a}}(1)(z)}\ d\mu_\nu(z) = e^{\frac{2\lambda}{\nu}} \int_{\mathbb{C}^d} e^{\langle z|b\rangle}\ e^{-\langle a|z\rangle}\ d\mu_\nu(z) = e^{\frac{\lambda}{\nu}}.$$

Therefore the eigenvalues of the *Berezin transform* $\mathcal{B} = \mathcal{T}'\mathcal{T}$, i.e.

$$\mathcal{B}(f)(z) = \int_{\mathbb{C}^d} \frac{f(w)\,|K^{(\nu)}(z,w)|^2}{K^{(\nu)}(z,z)}\ d\mu_\nu(w) = \int_{\mathbb{C}^d} f(w)\ e^{-\nu|z-w|^2}\ d\mu_0(w)$$

are given by

$$\tilde{\mathcal{B}}(\lambda) = e^{\frac{\lambda}{\nu}}, \quad \lambda \in \mathbb{C}.$$

From here one concludes that

$$\mathcal{B} = e^{\frac{\Delta}{\nu}}.$$

We remark that in the case of the Berezin transform one can compute the eigenvalues directly. Indeed, if $\lambda = \langle a, b\rangle$ then

$$\tilde{\mathcal{B}}(\lambda) = \mathcal{B}(e_{a,b})(0) = \int_{\mathbb{C}^d} e_{a,b}(w) d\mu_\nu(w) = \int_{\mathbb{C}^d} e^{\langle w|b\rangle + \langle a|w\rangle} d\mu_\nu(w) = e^{\frac{\lambda}{\nu}}.$$

We calculate now the standard functions of our theory in the case of the Toeplitz calculus $\mathcal{T}$. First

$$T_b(z,w) = \mathcal{T}_b(K_w)(z)/K_w(z) = \int_{\mathbb{C}^d} b(\xi) \frac{K(z,\xi)\ K(\xi,w)}{K(z,w)}\ d\mu_\nu(\xi).$$

Hence

$$\beta^{\mathcal{T}}(b)(z) = T_b(z,0) = \int_{\mathbb{C}^d} b(\xi) \frac{K(z,\xi)}{K(\xi,\xi)}\ d\mu_0(\xi)$$

and therefore

$$F^{\mathcal{T}}(z,\xi) = \frac{K(z,\xi)}{K(\xi,\xi)} = e^{\nu\langle z-\xi|\xi\rangle}\ .$$

From this we conclude that

$$T_\eta(K_w)(z) = \frac{K(z,\eta)\ K(\eta,w)}{K(\eta,\eta)},$$

and therefore $T_\eta f = \langle f, k_\eta\rangle_\nu k_\eta$ for all $f \in \mathcal{F}_\nu$. In particular $T_0 = k_0 \otimes k_0 = 1 \otimes 1$, i.e. $T_0 f = \langle f, 1\rangle 1$. This is the K-invariant operator which determines $\mathcal{T}$. Also, it follows that

$$\mathfrak{b}_{\mathcal{T}}(\xi) = \langle T_\xi k_0, k_0\rangle = |\langle k_\xi, 1\rangle|^2 = e^{-\nu|\xi|^2}, \quad \forall \xi \in \mathbb{C}^d.$$

Example 4.2: The Weyl calculus. It is easy to see that the symmetry at w is given by $s_w(z) = -z + 2w$. Therefore the corresponding isometry of $L^2(\mathbb{C}^d, \mu_\nu)$ and $\mathcal{F}_\nu$ is

$$U(s_w)(f)(z) = f(-z+2w)\ e^{-2\nu\,|w|^2 + 2\nu\,\langle z|w\rangle}. \tag{4.78}$$

The Weyl calculus is defined by

$$\mathcal{W}_b = \int_{\mathbb{C}^d} b(w)\ U(s_w)\ \widetilde{d\mu_0}(w),$$

where $d\mu_0(w) = 2^d \, d\tilde{\mu}_0(w) = \left(\frac{2\nu}{\pi}\right)^d \, dm(w)$. Namely,

$$\mathcal{W}_b(f)(z) = \left(\frac{2\nu}{\pi}\right)^d \int_{\mathbb{C}^d} b(w) \; f(-z+2w) \; e^{-2\nu\,|w|^2 + 2\nu\,\langle z|w\rangle} \; dm(w).$$

It follows that for $a, b \in \mathbb{C}^d$ with $\lambda = \langle a, b\rangle$

$$\begin{aligned} \mathcal{W}_{e_{a,b}}(1)(z) &= \left(\frac{2\nu}{\pi}\right)^d \int_{\mathbb{C}^d} e^{\langle w|b\rangle + \langle a|w\rangle + 2\nu\,\langle z|w\rangle} \; e^{-2\nu\,|w|^2} \; dm(w) \\ &= \int_{\mathbb{C}^d} e^{\langle w|b\rangle} \; K_{2\nu}(z + \frac{a}{2\nu}, \; w) \; d\mu_{2\nu}(w) = e^{\langle z + \frac{a}{2\nu}|b\rangle} = e^{\frac{\lambda}{2\nu}} \; e^{\langle z|b\rangle}. \end{aligned}$$

In particular, if $\lambda = \langle a|b\rangle$ then

$$\mathfrak{a}_{\mathcal{W}}(\lambda) = \mathcal{W}_{e_{a,b}}(1)(0) = e^{\frac{\lambda}{2\nu}}.$$

Similarly,

$$\mathcal{W}_{e_{-b,-a}}(1)(z) = e^{\frac{\overline{\lambda}}{2\nu}} \; e^{-\langle z|a\rangle},$$

and hence,

$$\begin{aligned} \widetilde{\mathcal{W}'\mathcal{W}}(\lambda) &= \int_{\mathbb{C}^d} \mathcal{W}_{e_{a,b}}(1)(z) \, \overline{\mathcal{W}_{e_{-b,-a}}(1)(z)} \; d\mu_\nu(z) \\ &= e^{\frac{\lambda}{\nu}} \int_{\mathbb{C}^d} e^{\langle z|b\rangle} \; e^{-\langle a|z\rangle} \; d\mu_\nu(z) = e^{\frac{\lambda}{\nu}} \; e^{-\frac{\langle a|b\rangle}{\nu}} = 1. \end{aligned}$$

Thus the eigenvalues of the link transform $\mathcal{W}'\mathcal{W}$ associated with the Weyl calculus are

$$\widetilde{\mathcal{W}'\mathcal{W}}(\lambda) \equiv 1, \qquad \forall\, \lambda \in \mathbb{C}.$$

From this we deduce that

$$\mathcal{W}'\mathcal{W} = I,$$

namely $\mathcal{W}$ is an isometry from $L^2(\mathbb{C}^d, \mu_\nu)$ into the Hilbert-Schmidt operators $\mathcal{S}_2(\mathcal{F}_\nu)$ on the weighted Fock space $\mathcal{F}_\nu = L^2_a(\mathbb{C}^d, \mu_\nu))$. This result is well-known, but its derivation by direct methods is more involved.

It is easy to compute the standard functions of our theory in the case of the Weyl calculus. First

$$\begin{aligned} W_b(z, w) &= \mathcal{W}_b(K_w)(z) / K_w(z) \\ &= e^{-2\nu\langle z|w\rangle} \int_{\mathbb{C}^d} b(\xi) \; \frac{e^{2\nu\,\langle z|\xi\rangle} \; e^{2\nu\,\langle \xi|w\rangle}}{e^{2\nu\,\langle \xi|\xi\rangle}} \; d\tilde{\mu}_0(\xi) \\ &= e^{-2\nu\langle z|w\rangle} \int_{\mathbb{C}^d} b(\xi) \; K^{(2\nu)}(z, \xi) \; K^{(2\nu)}(\xi, w) \; d\mu_{2\nu}(\xi). \end{aligned}$$

Hence

$$\beta^{\mathcal{W}}(b)(z) = W_b(z, 0) = \int_{\mathbb{C}^d} b(\xi) \; K^{(2\nu)}(z, \xi) \; d\mu_{2\nu}(\xi) = P^{(2\nu)}(b)(z).$$

Thus, with respect to the invariant measure $d\mu_0(\xi) = \left(\frac{2\nu}{\pi}\right) \, dm(\xi)$,

$$F^{\mathcal{W}}(z,\xi) = e^{2\nu\,\langle z|\xi\rangle - 2\nu\,|\xi|^2} = j(s_\xi, z).$$

Therefore we find easily that $W_\eta(K_w)(z) = K_w(s_\eta(z))\ j(s_\eta, z)$. Namely

$$W_\eta = U(s_\eta).$$

In particular, the K-invariant operator on $\mathcal{F}_\nu$ which determines $\mathcal{W}$ is

$$W_0 = U(s_0) = \sum_{k=0}^{\infty} (-1)^k\ P_k.$$

Moreover, using (4.78) we see that the fundamental function $\mathfrak{b}_{\mathcal{W}}(\xi)$ is given by

$$\mathfrak{b}_{\mathcal{W}}(\xi) = \langle W_\xi 1, 1\rangle = U(s_\xi(1)(0) = e^{-2\nu|\xi|^2}.$$

Example 4.3: Wick calculus. Let b be a real-analytic function with everywhere convergent Taylor expansion

$$b(z) = \sum_{\alpha,\beta\in\mathbb{N}^d} \frac{\partial^\alpha}{\alpha!}\,\frac{\overline{\partial}^\beta}{\beta!}\,b(0)\ z^\alpha \overline{z}^\beta.$$

We define the Wick calculus via

$$\mathcal{E}_b = \sum_{\alpha,\beta\in\mathbb{N}^d} \frac{\partial^\alpha}{\alpha!}\,\frac{\overline{\partial}^\beta}{\beta!}\,b(0)\ \mathcal{T}_{z^\alpha}\ \mathcal{T}^*_{z^\beta} \tag{4.79}$$

where $\mathcal{T}_{z^\alpha}$ is the Toeplitz operator with symbol z^α. Notice that this definition is consistent with the definition given in subsection 3.1. Thus, if $b(z) = \sum_j f_j(z)\ \overline{g_j(z)}$ with f_j, g_j entire holomorphic functions, then the sesqui-holomorphic extension of b is $E_b(z,w) = \sum_j\ f_j(z)\ \overline{g_j(w)}$, and $\mathcal{E}_b = \sum_j\ \mathcal{T}_{f_j}\ \mathcal{T}^*_{g_j}$. In particular, if we use the Taylor series of b, we see that

$$\beta^{\mathcal{E}}(b)(z) = E_b(z,0) = \sum_{\alpha\in\mathbb{N}^d} \frac{\partial^\alpha}{\alpha!} b(0)\ z^\alpha.$$

In the special case of the symbol $e_{a,b}(z) = e^{\langle z|b\rangle + \langle a|z\rangle}$ we obtain $\mathcal{E}_{e_{a,b}}(1)(z) = E_{e_{a,b}}(z,0) = e^{\langle z|b\rangle}$, and similarly $\mathcal{E}_{e_{-b,-a}}(1)(z) = e^{-\langle z|a\rangle}$. Hence, if $\langle a,b\rangle = \lambda$ then

$$\int_{\mathbb{C}^d} \mathcal{E}_{e_{a,b}}(1)(z)\,\overline{\mathcal{E}_{e_{-b,-a}}(1)(z)}\ d\mu_\nu(z) = \int_{\mathbb{C}^d} e^{\langle z|b\rangle}\ e^{-\langle a|z\rangle}\ d\mu_\nu(z) = e^{-\frac{\lambda}{\nu}}.$$

Thus the eigenvalues of the link transform associated with the Wick calculus are

$$\widetilde{\mathcal{E}'\mathcal{E}}(\lambda) = e^{-\frac{\lambda}{\nu}}.$$

4.2 The calculi associated with the k-homogeneous projections

Let $\mathcal{P}_k$ denote the subspace of $\mathcal{F}_\nu$ consisting of all homogeneous polynomials of degree k. It has a reproducing kernel $K_k^(z,w) = K_k^{(\nu)}(z,w)$, where

$$K_k(z,w) = K_k^{(\nu)}(z,w) = \frac{\nu^k \langle z|w\rangle^k}{k!}.$$

The orthogonal projection $P_k : \mathcal{F}_\nu \to \mathcal{P}_k$ is given by $P_k(f)(z) = \langle f,\ K_k(\cdot,z)\rangle_\nu$, and the covariant field of operators generated by P_k is $P_{k,\xi} = U(g_\xi)\ P_k\ U(g_\xi)^{-1}$.

Lemma 4.1 *For all $k \in \mathbb{N},\quad \xi \in \mathbb{C}^d$,*

$$\begin{aligned} P_{k,\xi}(K_w)(z) &= U(g_\xi) \otimes U(g_\xi)^*(K_k)(z,w) \\ &= j(g_\xi,z)\ K_k(z-\xi, w-\xi)\overline{j(g_\xi,w)} \\ &= e^{\nu\langle z|\xi\rangle}\ K_k(z-\xi,w-\xi)e^{\nu\langle\xi|w\rangle}\ e^{-\nu|\xi|^2}. \end{aligned} \tag{4.80}$$

Proof:

$$\begin{aligned} P_{k,\xi}(K_w)(z) &= P_k\,(U(g_{-\xi})\,K_w)(z-\xi)\ j(g_{-\xi},z) \\ &= \langle U(g_{-\xi})\,K_w,\ \ K_k(\cdot,z-\xi)\rangle_\nu\ j(g_{-\xi},z) \\ &= \langle K_w,\ \ U(g_\xi)\ K_k(\cdot,z-\xi)\rangle_\nu\ j(g_{-\xi},z) \\ &= K_k(z-\xi,\ \ w-\xi)\ j(g_{-\xi},z)\overline{j(g_{-\xi},w)}. \end{aligned}$$

Q.E.D.

Lemma 4.2 *For any $\ell, j, n \in \mathbb{N}$ and $a, b \in \mathbb{C}^d$*

$$\int_{\mathbb{C}^d} |z|^{2\ell}\ \langle z|b\rangle^j\ \langle a|z\rangle^n\ d\mu_\nu(z) = \delta_{j,n}\ \frac{\langle a|b\rangle^j}{\nu^{\ell+j}}\ \frac{(d+\ell+j-1)!\,j!}{(d+j-1)!}.$$

Proof: Integrating in polar coordinates (where $S = \{\xi \in \mathbb{C}^d;\ \ |\xi| = 1\}$ and σ is the $U(d)$-invariant probability measure on S) we find

$$\begin{aligned} \int_{\mathbb{C}^d} |z|^{2\ell}\ \langle z|b\rangle^j\ \langle a|z\rangle^n\ d\mu_\nu(z) &= \\ &= \frac{2\nu^d}{(d-1)!}\int_0^\infty r^{2d-1}\ r^{2\ell}\ r^{j+n}\ e^{-\nu r^2}dr\ \int_S \langle\xi|b\rangle^j\ \langle a|\xi\rangle^n\ d\sigma(\xi) \\ &= \delta_{j,n}\ \frac{\nu^d}{(d-1)!}\int_0^\infty t^{d+\ell+j-1}\ e^{-\nu\, t}\ dt\ \ \frac{j!\,\langle a,b\rangle^j}{(d)_j} \\ &= \delta_{j,n}\ \frac{(d+\ell+j-1)!\,j!}{(d-1)!(d)_j}\ \frac{\langle a|b\rangle^j}{\nu^{\ell+j}}. \end{aligned}$$

Q.E.D.

Theorem 4.3 *Let $k, \ell \in \mathbb{N}$ and write $\mathcal{B}^{k,\ell} := (\mathcal{A}^{P_\ell})' \mathcal{A}^{P_k}$. Then the eigenvalues of $\mathcal{B}^{k,\ell}$ are given by*

$$\widetilde{\mathcal{B}^{k,\ell}}(\lambda) = e^{\frac{\lambda}{\nu}}\ q_k(\frac{\lambda}{\nu})\ q_\ell(\frac{\lambda}{\nu}), \quad \forall \lambda \in \mathbb{C},$$

where

$$q_k(x) = \sum_{j=0}^{k} \binom{d+k-1}{d+j-1} \frac{x^j}{j!}. \tag{4.81}$$

Proof: Let $\lambda \in \mathbb{C}$ and let $a, b \in \mathbb{C}^d$ be so that $\langle a|b\rangle = \lambda$. Then, using Lemma 4.1, we have

$$\begin{aligned}
\tilde{\mathcal{B}}^{k,\ell}(\lambda) &= \left((\mathcal{A}^{P_\ell})'\mathcal{A}^{P_k}\ e_{a,b}\right)(0) = \langle \mathcal{A}^{P_k}_{e_{a,b}},\ P_\ell\rangle_{S_2} = \int_{\mathbb{C}^d} e_{a,b}(\xi)\ \langle P_{k,\xi},\ P_\ell\rangle_{S_2}\ d\mu_0(\xi) \\
&= \int_{\mathbb{C}^d} e_{a,b}(\xi) \left(\iint_{\mathbb{C}^d\times\mathbb{C}^d} P_{k,\xi}(K_w)(z)\ \overline{P_\ell(K_w)(z)}\ d\mu_\nu(z)\ d\mu_\nu(w) \right) d\mu_0(\xi) \\
&= \int_{\mathbb{C}^d} e^{\langle a|\xi\rangle}\ e^{\langle \xi|b\rangle} \left(\iint_{\mathbb{C}^d\times\mathbb{C}^d} e^{\nu\,\langle z|\xi\rangle}\ K_k(z-\xi,\ w-\xi)\ e^{\nu\,\langle \xi|w\rangle} \right. \\
&\qquad\qquad \left. K_\ell(w,z)\ d\mu_\nu(z)\ d\mu_\nu(w) \right) d\mu_\nu(\xi) \\
&= \int_{\mathbb{C}^d} e^{\langle a|\xi\rangle}\ e^{\langle \xi|b\rangle} \left(\iint_{\mathbb{C}^d\times\mathbb{C}^d} K_k(z, w-\xi) K_\ell(w, z+\xi) e^{-\nu\,\langle \xi|z\rangle}\ d\mu_\nu(z) \right. \\
&\qquad\qquad \left. e^{\nu\,\langle \xi|w\rangle}\ d\mu_\nu(w) \right) d\mu_\nu(\xi).
\end{aligned}$$

Interchanging the order of integration, the last integral becomes

$$\begin{aligned}
&\iint_{\mathbb{C}^d\times\mathbb{C}^d} \left[\int_{\mathbb{C}^d} \left\{ K_k(z, w-\xi)\ K_\ell(w, z+\xi)\ e^{\langle a|\xi\rangle} \right\}\ e^{\nu\,\langle \xi|w-z+\frac{b}{\nu}\rangle}\ d\mu_\nu(\xi) \right] d\mu_\nu(z)\, d\mu_\nu(w) \\
&\quad = \iint_{\mathbb{C}^d\times\mathbb{C}^d} K_k\left(z, z-\frac{b}{\nu}\right)\ K_\ell\left(w, w+\frac{b}{\nu}\right)\ e^{\langle a|w-z+\frac{b}{\nu}\rangle}\ d\mu_\nu(z)\ d\mu_\nu(w) \\
&\quad = e^{\frac{\lambda}{\nu}}\ I_k(a,b)\ I_\ell(a,b),
\end{aligned}$$

where

$$I_k(a,b) := \int_{\mathbb{C}^d} K_k\left(z, z+\frac{b}{\nu}\right)\ e^{\langle a|z\rangle}\ d\mu_\nu(z)$$

and we used the fact that $I_k(a,b) = I_k(-a,-b)$. To calculate $I_k(a,b)$ we expand $K_k(z, z+\frac{a}{\nu})$ and use Lemma 4.2,

$$\begin{aligned} I_k(a,b) &= \frac{\nu^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \int_{\mathbb{C}^d} |z|^{2(k-j)} \frac{\langle z|b\rangle^j}{\nu^j} e^{\langle a|z\rangle} \, d\mu_\nu(z) \\ &= \nu^k \sum_{j=0}^{k} \frac{\nu^{-j}}{j!(k-j)!} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{C}^d} |z|^{2(k-j)} \langle z|b\rangle^j \, \langle a|z\rangle^n \, d\mu_\nu(z) \\ &= \nu^k \sum_{j=0}^{k} \frac{\nu^{-j}}{(j!)^2(k-j)!} \frac{\lambda^j}{\nu^k} \frac{(d+k-1)!}{(d+j-1)!} j! \\ &= \sum_{j=0}^{k} \binom{d+k-1}{d+j-1} \frac{(\frac{\lambda}{\nu})^j}{j!} = q_k\left(\frac{\lambda}{\nu}\right). \end{aligned}$$

Q.E.D.

Remark 4.1 *Notice that the polynomial q_k can be written also as*

$$q_k(x) = \sum_{j=0}^{k} \frac{(d+j)_{k-j}}{(k-j)!} \frac{x^j}{j!}$$

and that

$$q_k(x) = \binom{k+d-1}{d-1} \sum_{j=0}^{k} \frac{(-k)_j}{(d)_j} \frac{(-x)^j}{j!} = \binom{k+d-1}{d-1} \, {}_1F_1(-k; d; -x).$$

Applying Theorem 4.3 we obtain the following result.

Theorem 4.4 *Let $T = \sum_{k=0}^{\infty} t_k P_k$, $S = \sum_{\ell=0}^{\infty} s_\ell P_\ell$ be K-invariant operators on holomorphic functions and let $\mathcal{A}^T$, $\mathcal{A}^S$ be the associated calculi. Let $\mathcal{B}^{T,S} = (\mathcal{A}^S)' \mathcal{A}^T$ be the corresponding link transform. Then*

$$\widetilde{\mathcal{B}^{T,S}}(\lambda) = e^{\frac{\lambda}{\nu}} \sum_{k=0}^{\infty} t_k \, q_k(\frac{\lambda}{\nu}) \sum_{\ell=0}^{\infty} \overline{s_\ell} \, q_\ell(\frac{\lambda}{\nu}).$$

Proof: We have

$$\begin{aligned} \widetilde{\mathcal{B}^{T,S}}(\lambda) &= \sum_{k,\ell \geq 0} t_k \, \overline{s_\ell} \, \widetilde{\mathcal{B}^{k,\ell}}(\lambda) \\ &= \sum_{k,\ell \geq 0} t_k \, \overline{s_\ell} \, e^{\frac{\lambda}{\nu}} \, q_k(\frac{\lambda}{\nu}) \, q_\ell(\frac{\lambda}{\nu}) = e^{\frac{\lambda}{\nu}} \sum_{k \geq 0} t_k \, q_k(\frac{\lambda}{\nu}) \sum_{\ell \geq 0} \overline{s_\ell} \, q_\ell(\frac{\lambda}{\nu}). \end{aligned}$$

Q.E.D.

The following example of a one-parameter family of calculi will play an important role in the sequel.

Example 4.4: Let $\alpha \in \mathbb{C}$, $\alpha \neq 1$ and set

$$t_k(\alpha) = (1-\alpha)^d \, \alpha^k \quad \text{and} \quad T(\alpha) = \sum_{k=0}^{\infty} t_k(\alpha) \, P_k,$$

i.e. $(T(\alpha) f)(z) = (1-\alpha)^d f(\alpha z)$. Let $\mathcal{A}^\alpha = \mathcal{A}^{T(\alpha)}$ be the associated calculus. Then Theorem 4.3 shows that the eigenvalues of $\mathcal{B}^{\alpha,\beta} = \mathcal{B}^{T(\alpha),T(\beta)}$ are given by

$$\widetilde{\mathcal{B}^{\alpha,\beta}}(\lambda) = e^{\frac{\lambda}{\nu}} \sum_{k\geq 0} \alpha^k \, q_k(\frac{\lambda}{\nu}) \sum_{\ell \geq 0} \overline{\beta^\ell} \, q_\ell(\frac{\lambda}{\nu}) \; (1-\alpha)^d (1-\overline{\beta})^d.$$

Claim: If $|\alpha| < 1$ then

$$\sum_{k=0}^{\infty} \alpha^k \, q_k(x) = \frac{1}{(1-\alpha)^d} \, \exp\left(\frac{\alpha}{1-\alpha}\, x\right). \tag{4.82}$$

Indeed, by absolute convergence we can interchange the order of summation and obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \alpha^k \, q_k(x) &= \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{k=j}^{\infty} \frac{(d+j)_{k-j}}{(k-j)!} \, \alpha^k = \sum_{j=0}^{\infty} \frac{(\alpha x)^j}{j!} \sum_{\ell=0}^{\infty} (d+j)_\ell \, \frac{\alpha^\ell}{\ell!} \\ &= \sum_{j=0}^{\infty} \frac{(\alpha x)^j}{j!} \, (1-\alpha)^{-(d+j)} = \frac{1}{(1-\alpha)^d} \sum_{j=0}^{\infty} \frac{\left(\frac{\alpha x}{1-\alpha}\right)^j}{j!} \\ &= \frac{1}{(1-\alpha)^d} \exp\left(\frac{\alpha x}{1-\alpha}\right). \end{aligned}$$

It follows that

$$\sum_{k=0}^{\infty} t_k(\alpha) \; q_k\,(\frac{\lambda}{\nu}) = \exp\left(\frac{\alpha}{1-\alpha}\,\frac{\lambda}{\nu}\right).$$

If also $|\beta| < 1$, then

$$\widetilde{\mathcal{B}^{\alpha,\beta}}(\lambda) = \exp\left\{\frac{1-\alpha\,\overline{\beta}}{(1-\alpha)(1-\overline{\beta})}\,\frac{\lambda}{\nu}\right\}, \tag{4.83}$$

and in particular,

$$\widetilde{\mathcal{B}^{\alpha,\alpha}}(\lambda) = \exp\left\{\frac{1-|\alpha|^2}{|1-\alpha|^2}\,\frac{\lambda}{\nu}\right\}. \tag{4.84}$$

By analytic continuation in $\alpha, \overline{\beta}$ we obtain:

Proposition 4.1 *Let $\alpha, \beta \in \mathbb{C} \setminus \{1\}$. Then the eigenvalues of $\mathcal{B}^{\alpha,\beta}$ are given by* (4.83).

Corollary 4.1 *Let $\alpha, \beta \in \mathbb{C} \setminus \{1\}$ be so that $1 - \alpha\overline{\beta} = (1-\alpha)(1-\overline{\beta})$ (i.e. $\alpha + \overline{\beta} = 2\,\alpha\,\overline{\beta}$). Then $\mathcal{B}^{\alpha,\beta}$ coincides with the Berezin transform (namely, with the link transform of the Toeplitz calculus).*

Proof: We have $\widetilde{\mathcal{B}^{\alpha,\beta}}(\lambda) = e^{\frac{\lambda}{\nu}}, \;\; \forall\, \lambda \in \mathbb{C}$. Since this is true for the Berezin transform $\mathcal{B}$ we conclude that $\mathcal{B}^{\alpha,\beta} = \mathcal{B}$. Q.E.D.

Corollary 4.2 *Let $\alpha, \beta \in \mathbb{C} \setminus \{1\}$ be so that $\alpha\,\overline{\beta} = 1$. Then $\mathcal{B}^{\alpha,\beta} = I$.*

Proof: We have $\widetilde{\mathcal{B}^{\alpha,\beta}}(\lambda) = 1, \;\; \forall\, \lambda \in \mathbb{C}$. Q.E.D.

Corollary 4.3 *Let $\alpha \in \mathbb{C} \setminus \{1\}$ be so that $|\alpha| = 1$. Then $\mathcal{B}^{\alpha,\alpha} = I$. Consequently $\mathcal{A}^\alpha$ is an isometry.*

Proposition 4.2 *Let $\alpha \in \mathbb{C} \setminus \{1\}$ and let $\tau = \frac{1-|\alpha|^2}{|1-\alpha|^2} > 0$. Then $\mathcal{B}^{\alpha,\alpha} f = f * \mu_{\frac{\nu}{\tau}}$, i.e.*

$$(\mathcal{B}^{\alpha,\alpha} f)(z) = \int_{\mathbb{C}^d} f(g(w))\ d\mu_{\frac{\nu}{\tau}}(w).$$

Proof: The operator of convolution with $\mu_{\frac{\nu}{\tau}}$ is the Berezin transform for the Toeplitz calculus associated with $L^2_a(\mathbb{C}^d, \mu_{\frac{\nu}{\tau}})$. Hence its eigenvalues are $e^{\lambda/(\frac{\nu}{\tau})} = e^{\tau\frac{\lambda}{\nu}} = \widetilde{\mathcal{B}^{\alpha,\alpha}}(\lambda)$.

Q.E.D.

The following general fact follows by elementary properties of intertwining operators.

Lemma 4.3 *Let $\mathcal{A}$ be an invariant symbolic calculus, considered formally as a map between Hilbert spaces $\mathcal{A}: L^2(D, \mu_0) \to S_2(\mathcal{H})$. Let*

$$\mathcal{A} = \mathcal{U}\,\mathcal{B}^{\frac{1}{2}} \quad (\mathcal{B} = \mathcal{A}'\,\mathcal{A}), \quad \text{and} \quad \mathcal{A} = \mathcal{Q}^{\frac{1}{2}}\mathcal{V} \quad (\mathcal{Q} = \mathcal{A}\,\mathcal{A}') \tag{4.85}$$

be the polar decompositions of $\mathcal{A}$, with $\mathcal{U}$, $\mathcal{V}$ minimal partial isometries (i.e. $Ker\,(\mathcal{U}) = ker\,(\mathcal{A})$ and $ker\,(\mathcal{V}) = ker\,(\mathcal{A}')$). Then

$$\mathcal{B}^{\frac{1}{2}}(f \circ g) = (\mathcal{B}^{\frac{1}{2}} f) \circ g, \quad \forall\, g \in G, \quad \forall\, f \in Dom(\mathcal{B}^{\frac{1}{2}}). \tag{4.86}$$

$$\mathcal{Q}^{\frac{1}{2}}(\pi(g)(X)) = \pi(g)(\mathcal{Q}^{\frac{1}{2}}(X)), \quad \forall\, g \in G, \quad \forall\, X \in Dom(\mathcal{Q}^{\frac{1}{2}}), \tag{4.87}$$

where

$$\pi(g)(X) := U(g)\ X\ U(g)^{-1}, \quad \forall\, g \in G.$$

$\mathcal{U}$ and $\mathcal{V}$ are intertwining operators, i.e.

$$\pi(g)(\mathcal{U}_b) = \mathcal{U}_{b\circ g^{-1}}, \quad \pi(g)(\mathcal{V}_b) = \mathcal{V}_{b\circ g^{-1}}$$

for all $g \in G$ and b in the appropriate domains. Thus, $\mathcal{U}$ and $\mathcal{V}$ are invariant symbolic calculi.

Proof: (4.86) and (4.87) follow immediately from Proposition 2.7. Notice also that

$$\mathcal{A}'(\pi(g)\,X) = \mathcal{A}'(X) \circ g^{-1}, \quad \forall g \in G,\ \forall X \in \mathrm{Dom}(\mathcal{A}'), \tag{4.88}$$

The fact that

$$\pi(g)(\mathcal{A}_b) = \mathcal{A}_{b\circ g^{-1}}\ , \quad \forall\, g \in G, \quad \forall\, X \in \mathrm{Dom}(\mathcal{A}),$$

clearly implies

$$\pi(g)(\mathcal{U}_f) = \mathcal{U}_{f\circ g^{-1}}\ , \quad \forall\, g \in G, \quad \forall\, f \in Ran\,(\mathcal{B}^{\frac{1}{2}}).$$

Hence, $\mathcal{U}$ is an invariant symbolic calculus with $\mathrm{Dom}(\mathcal{U}) = \mathrm{Ran}(\mathcal{B}^{\frac{1}{2}})$. Similarly, using the fact that $\mathcal{Q}^{\frac{1}{2}}$ is one-to-one on the range of $\mathcal{V}$, we see that $\pi(g)(\mathcal{V}_b) = \mathcal{V}_{b\circ g^{-1}}$ for all $g \in G$ and all $b \in \mathrm{Dom}(\mathcal{V})$.

Q.E.D.

Theorem 4.5 *Let $k \in \mathbb{N}$ and let $\mathcal{A}^{P_k} = \mathcal{U}^{(k)}\,|\mathcal{A}^{P_k}|$, where $|\mathcal{A}^{P_k}| = (\mathcal{A}^{P_k\prime}\,\mathcal{A}^{P_k})^{\frac{1}{2}} = \mathcal{B}^{(k)\,\frac{1}{2}}$ be the polar decomposition* (4.85) of the invariant symbolic calculus $\mathcal{A}^{P_k}$ associated with the projection P_k. Then

$$\widetilde{(\mathcal{B}^{(k)})^{\frac{1}{2}}}(\lambda) = e^{\frac{\lambda}{2\nu}}\ |q_k(\frac{\lambda}{\nu})|, \quad \forall\, \lambda \in (-\infty, 0], \ \text{ and } \ (\mathcal{B}^{(k)})^{\frac{1}{2}} = e^{\frac{\Delta}{2\nu}}\ |q_k|(\frac{\Delta}{\nu}). \tag{4.89}$$

as an operator on $L^2(\mathbb{C}^d, \mu_0) \equiv L^2(\mathbb{R}^{2d}, m)$. Moreover

$$\mathcal{U}^{(k)} = \mathcal{W}\, V_k, \quad \text{where } V_k := \operatorname{sgn}(q_k)(\frac{\Delta}{\nu}) \tag{4.90}$$

and $\mathcal{W}$ is the Weyl calculus.

Proof: Let $a, b \in \mathbb{C}^d$ and denote $\lambda = \langle a|b\rangle$. We claim that for all $z, w \in \mathbb{C}^d$

$$\begin{aligned} A^{(k)}_{e_{a,b}}(z,w) &= \mathcal{A}^{(P_k)}_{e_{a,b}}(K_w)(z)/K_w(z) \\ &= \binom{d-1+k}{d-1}\ {}_1F_1\left(d+k;\ d;\ \frac{\lambda}{\nu}\right)\ e^{\langle z|b\rangle}\ e^{\langle a|w\rangle}. \end{aligned}$$

Here we use the standard notation ${}_1F_1\,(\alpha;\ \beta;\ x) = \sum\limits_{n\geq 0} \frac{(\alpha)_n}{(\beta)_n}\frac{x^n}{n!}$. Since $A^{(k)}_{e_{a,b}}(z,w)$ is holomorphic in a and z and anti-holomorphic in b and w, it suffices to consider $a = b$ and $z = w$. In this case, we obtain from Lemma 4.1

$$\begin{aligned} \mathcal{A}^{P_k}_{e_{a,b}}(K_z)(z) &= \int\limits_{\mathbb{C}^d} e^{\langle a|\xi\rangle}\ e^{\langle \xi|a\rangle}\ e^{\nu\,\langle z|\xi\rangle}\ K_k\,(z-\xi,\ z-\xi)\ e^{\nu\,\langle \xi|z\rangle}\ d\mu_\nu(\xi) \\ &= \frac{\nu^k}{k!}\int\limits_{\mathbb{C}^d} e^{\nu\,\langle \eta+z|z+\frac{a}{\nu}\rangle}\ e^{\nu\,\langle z+\frac{a}{\nu}|\eta+z\rangle}\ |\eta|^{2k}\ d\mu_\nu\,(\eta+z) \\ &= \frac{\nu^k}{k!}\ e^{\nu\,|z|^2}\ e^{\langle z|a\rangle+\langle a|z\rangle}\int\limits_{\mathbb{C}^d} e^{\langle \eta|a\rangle}\ e^{\langle a|\eta\rangle}\ |\eta|^{2k}\ d\mu_\nu\,(\eta) \\ &= \frac{\nu^k}{k!}\ e^{\nu\,|z|^2}\ e^{\langle z|a\rangle+\langle a|z\rangle}\sum_{m,n=0}^{\infty}\ \int\limits_{\mathbb{C}^d}\frac{\langle \eta|a\rangle^m}{m!}\ \frac{\langle a|\eta\rangle^n}{n!}\ |\eta|^{2k}\ d\mu_\nu\,(\eta). \end{aligned}$$

The last integral is evaluated by Lemma 4.2

$$\int\limits_{\mathbb{C}^d}\frac{\langle \eta|a\rangle^m}{m!}\frac{\langle a|\eta\rangle^n}{n!}|\eta|^{2k}\ d\mu_\nu(\eta) = \delta_{m,n}\frac{(d+k+n-1)!}{(d-1)!\,(d)_n\,\nu^{k+n}}\frac{|a|^{2n}}{n!}. \tag{4.91}$$

Hence

$$\begin{aligned} \mathcal{A}^{P_k}_{e_{a,b}}(K_z)(z) &= \binom{d+k-1}{d-1}\sum_{n=0}^{\infty}\frac{(d+k)_n}{(d)_n}\ \frac{|a|^{2n}}{n!\,\nu^n}e^{\langle z|a\rangle}\ e^{\langle a|z\rangle}\ e^{\nu\,|z|^2} \\ &= \binom{d+k-1}{d-1}\ {}_1F_1\left(d+k;\ d;\ \frac{|a|^2}{\nu}\right)\ e^{\langle z|a\rangle}\ e^{\langle a|z\rangle}\ e^{\nu\,|z|^2}. \end{aligned}$$

Since a sesqui-holomorphic function is determined by its value on the "diagonal", we see that

$$\mathcal{A}^{P_k}_{e_{a,b}}(K_w)(z) = \binom{d+k-1}{d-1}\ {}_1F_1\left(d+k;\ d;\ \frac{\langle a|b\rangle}{\nu}\right)\ e^{\langle z|b\rangle}\ e^{\langle a|w\rangle}\ e^{\nu\,\langle z|w\rangle},$$

and (4.91) is established.

Lemma 4.4 *For all $x \in \mathbb{C}$ and $k \in \mathbb{N}$*

$$e^x\, q_k(x) = \binom{d+k-1}{d-1}\ {}_1F_1\,(d+k;\ d;\ x).$$

Proof: Interchanging the order of summation, we get

$$e^x\, q_k(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{j=0}^{k} \binom{d+k-1}{d+j-1} \frac{x^j}{j!} = \sum_{m=0}^{\infty} \sum_{j=0}^{k\wedge m} \binom{m}{j} \binom{d+k-1}{k-j} \frac{x^m}{m!}.$$

An easy combinatorial argument yields

$$\sum_{j=0}^{k\wedge m} \binom{m}{j} \binom{d+k-1}{k-j} = \binom{d+k+m-1}{k},$$

and thus

$$\begin{aligned} e^x\, q_k(x) &= \sum_{m=0}^{\infty} \binom{d+k+m-1}{k} \frac{x^m}{m!} \\ &= \sum_{j=0}^{k} \binom{d+k-1}{d-1} \sum_{m=0}^{\infty} \frac{(d+k)_m}{(d)_m} \frac{x^m}{m!} \\ &= \binom{d+k-1}{d-1}\ {}_1F_1\,(d+k;\ d;\ x). \end{aligned}$$

Q.E.D.

It follows from (4.91) and Lemma 4.4 that

$$A^{P_k}_{e_{a,b}}(z,w) = e^{\frac{\lambda}{\nu}} q_k\left(\frac{\lambda}{\nu}\right)\ e^{\langle z|b\rangle}\ e^{\langle a|w\rangle}. \tag{4.92}$$

Next, we know that

$$\widetilde{\mathcal{B}^{(k)}}(\lambda) = \widetilde{(\mathcal{A}^{P_k})'\,\mathcal{A}^{P_k}}(\lambda) = e^{\frac{\lambda}{\nu}}\ q_k\left(\frac{\lambda}{\nu}\right)^2, \quad \forall\, \lambda \in \mathbb{C}.$$

Considering $\mathcal{B}^{(k)\frac{1}{2}}$ as an operator on $L^2(\mathbb{C}^d, \mu_0)$, and knowing that the Plancherel measure is supported on $(-\infty, 0]$ (i.e. on the $e_{a,b}$ with $b = -a$), we conclude from

$$\widetilde{\mathcal{B}^{(k)}}^{\frac{1}{2}}(\lambda) = e^{\frac{\lambda}{2\nu}}\ |q_k\left(\frac{\lambda}{\nu}\right)|, \quad \forall\, \lambda \in (-\infty, 0]$$

that

$$\widetilde{\mathcal{B}^{(k)}}^{\frac{1}{2}} = e^{\frac{\Delta}{2\nu}}\ |q_k|(\frac{\Delta}{\nu}).$$

This establishes (4.89). Computing $A^{P_k}_{e_{a,b}}(z,w)$ via the factorization $\mathcal{A}^{P_k} = \mathcal{U}^{(k)}\mathcal{B}^{(k)\frac{1}{2}}$, we obtain

$$A^{P_k}_{e_{a,b}}(z,w) = e^{\frac{\lambda}{2\nu}}\ |q_k(\frac{\lambda}{\nu})|\ \frac{\mathcal{U}^{(k)}_{e_{a,b}}(K_w)(z)}{K_w(z)}.$$

Comparing this with (4.92) we conclude that

$$\frac{\mathcal{U}^{(k)}_{e_{a,b}}(K_w)(z)}{K_w(z)} = e^{\frac{\lambda}{2\nu}}\ \operatorname{sgn}(q_k(\frac{\lambda}{\nu}))\ e^{\langle z|b\rangle}\ e^{\langle a|w\rangle}. \tag{4.93}$$

However, the Weyl transform $\mathcal{W}$ satisfies

$$\begin{aligned}
\mathcal{W}_{e_{a,b}}(K_w)(z) &= 2^d \int_{\mathbb{C}^d} e^{\langle\xi|b\rangle}\ e^{\langle a|\xi\rangle}\ U(s_\xi)(K_w)(z)\ d\mu_0(\xi) \\
&= \int_{\mathbb{C}^d} e^{\langle\xi|b\rangle}\ e^{\langle a|\xi\rangle}\ e^{\nu\langle -z+2\xi|w\rangle}\ e^{2\nu\langle z|\xi\rangle}\ d\mu_{2\nu}(\xi) \\
&= \int_{\mathbb{C}^d} e^{\langle\xi|b+2\nu w\rangle}\ e^{2\nu\langle z+\frac{a}{2\nu}|\xi\rangle}\ d\mu_{2\nu}(\xi)\ e^{-\nu\langle z|w\rangle} \\
&= e^{\frac{\lambda}{2\nu}}\ e^{\langle z|b\rangle}\ e^{\langle a|w\rangle}\ e^{\nu\langle z|w\rangle},
\end{aligned}$$

i.e.

$$W_{e_{a,b}}(z,w) = \frac{\mathcal{W}_{e_{a,b}}(K_w)(z)}{K_w(z)} = e^{\frac{\lambda}{2\nu}}\ e^{\langle z|b\rangle}\ e^{\langle a|w\rangle}. \tag{4.94}$$

Comparing (4.93) and (4.94) we conclude that

$$\mathcal{U}^{(k)}_{e_{a,b}} = \mathcal{W}_{V_k(e_{a,b})}, \quad \text{where } V_k := \operatorname{sgn}(q_k)(\frac{\Delta}{\nu}).$$

Standard approximation arguments yield that $\mathcal{U}^{(k)}_f = \mathcal{W}_{V_k(f)}$ for all admissible symbols f. Hence (4.90) is established, and the proof of Theorem 4.5 is complete. Q.E.D

Theorem 4.6 *Let $\alpha \in \mathbb{C}\setminus\{1\}$ and let $\mathcal{A}^\alpha = \mathcal{U}^{(\alpha)}|\mathcal{A}^\alpha|$ be the polar decomposition of $\mathcal{A}^\alpha$, where $|\mathcal{A}^\alpha| = (\mathcal{B}^{\alpha,\alpha})^{\frac{1}{2}}$. Let $\alpha = \frac{z-1}{z+1}$ where $z = x+iy$. Then $|\mathcal{A}^\alpha| = \mathcal{B}^{\frac{x}{2}}$, where $\mathcal{B}$ is the Berezin transform associated with the Toeplitz calculus, and $\mathcal{U}^{(\alpha)} = \mathcal{A}^\beta$, where $\beta = \frac{iy-1}{iy+1}$.*

Proof: We know by (4.84) that

$$\widetilde{|\mathcal{A}^\alpha|}(\lambda) = \exp\left\{\frac{1}{2}\,\frac{1-|\alpha|^2}{|1-\alpha|^2}\,\frac{\lambda}{\nu}\right\} = \exp\left\{\frac{\lambda}{2\nu}\,x\right\}.$$

Also, $\tilde{\mathcal{B}}(\lambda) = e^{\frac{\lambda}{\nu}}$. Hence $\widetilde{|\mathcal{A}^\alpha|}(\lambda) = \widetilde{\mathcal{B}^{\frac{x}{2}}}(\lambda)$, and therefore $|\mathcal{A}^\alpha| = \mathcal{B}^{\frac{x}{2}}$. Next, using (4.82) and (4.92) we see that for any $a,b\in\mathbb{C}^d$ with $\langle a|b\rangle = \lambda$ and any $z,w\in\mathbb{C}^d$ we have for $|\alpha|<1$

$$\frac{\mathcal{A}^\alpha_{e_{a,b}}(K_w)(z)}{K_w(z)\ e^{\langle a|w\rangle}\ e^{\langle z|b\rangle}} = (1-\alpha)^d \sum_{k\ge 0} \alpha^k\ q_k(\frac{\lambda}{\nu}) e^{\frac{\lambda}{\nu}} = \exp\left\{\frac{1}{1-\alpha}\,\frac{\lambda}{\nu}\right\}.$$

By analytic continuation we conclude that

$$\frac{\mathcal{A}^\alpha_{e_{a,b}}(K_w)(z)}{K_w(z)\ e^{\langle a|w\rangle}\ e^{\langle z|b\rangle}} = \exp\left\{\frac{1}{1-\alpha}\,\frac{\lambda}{\nu}\right\}, \qquad \forall\ \alpha\in\mathbb{C}\setminus\{1\}. \tag{4.95}$$

Since

$$|\mathcal{A}^\alpha|(e_{a,b}) = \left\{\frac{1}{2}\,\frac{1-|\alpha|^2}{|1-\alpha|^2}\,\frac{\lambda}{\nu}\right\} e_{a,b},$$

we obtain also

$$\frac{\mathcal{A}^\alpha_{e_{a,b}}(K_w)(z)}{K_w(z)\ e^{\langle a|w\rangle}\ e^{\langle z|b\rangle}} = \exp\left\{\frac{1}{2}\,\frac{1-|\alpha|^2}{|1-\alpha|^2}\,\frac{\lambda}{\nu}\right\}\ \frac{\mathcal{U}^{(\alpha)}_{e_{a,b}}(K_w)(z)}{K_w(z)\ e^{\langle a|w\rangle}\ e^{\langle z|b\rangle}}.$$

Therefore, by (4.95), we obtain

$$\frac{\mathcal{U}^{(\alpha)}_{e_{a,b}}(K_w)(z)}{K_w(z)\ e^{\langle a|w\rangle}\ e^{\langle z|b\rangle}} = \exp\left\{\left(\frac{1}{1-\alpha} - \frac{1}{2}\,\frac{1-|\alpha|^2}{|1-\alpha|^2}\right)\frac{\lambda}{\nu}\right\} = \exp\left\{\frac{1}{1-\beta}\frac{\lambda}{\nu}\right\} \tag{4.96}$$

with

$$\beta = -\frac{1-2\alpha+|\alpha|^2}{1-2\bar{\alpha}+|\alpha|^2} = \frac{iy-1}{iy+1}.$$

Comparing (4.96) and (4.95) we conclude that $\mathcal{U}^{(\alpha)} = \mathcal{A}^{\beta}$. Q.E.D.

Remark 4.2 *In general, $\mathcal{U}^{(\alpha)} = \mathcal{A}^{\beta}$ is different from the Weyl calculus $\mathcal{W}$, and*

$$\mathcal{U}^{(\alpha)} = \mathcal{W} \iff \beta = -1 \iff \alpha \in \mathbb{R}\setminus\{1\} \iff z \in \mathbb{R}.$$

Remark 4.3 *Theorem 4.6 suggests another possible definition of the Wick calculus* (4.79), namely

$$\mathcal{E} = \mathcal{W}\ \mathcal{B}^{-\frac{1}{2}} \tag{4.97}$$

where $\mathcal{B}$ is the Berezin transform and $\mathcal{W}$ is the Weyl calculus.

Indeed, using (4.94) we see that for all $a, b \in \mathbb{C}^d$ with $\lambda = \langle a|b\rangle$ and all $z, w \in \mathbb{C}^d$ we have

$$\frac{\left(\mathcal{W}\ \mathcal{B}^{-\frac{1}{2}}\right)_{e_{a,b}}(K_w)(z)}{K_w(z)\ e^{\langle a|w\rangle}\ e^{\langle z|b\rangle}} = e^{-\frac{\lambda}{2\nu}}\ \frac{\mathcal{W}_{e_{a,b}}(K_w)(z)}{K_w(z)\ e^{\langle a|w\rangle}\ e^{\langle z|b\rangle}} = 1.$$

On the other hand, using the definition (4.79) we see that $\mathcal{E}_{e_{a,b}} = \mathcal{T}_{e_{0,b}}\ \mathcal{T}^*_{e_{0,a}}$, and therefore

$$\frac{\mathcal{E}_{e_{a,b}}(K_w)(z)}{K_w(z)\ e^{\langle a|w\rangle}\ e^{\langle z|b\rangle}} = \frac{e^{\langle z|b\rangle}\ \mathcal{T}^*_{e_{0,a}}\ (K_w)(z)}{K_w(z)\ e^{\langle a|w\rangle}\ e^{\langle z|b\rangle}} = 1,$$

since for holomorphic symbols φ we have $\mathcal{T}^*_{\varphi}(K_w) = \overline{\varphi(w)}\ K_w$. Therefore (4.97) holds.

Remark 4.4 *Theorem 4.6 suggests also to consider $\mathcal{E}$ as the limiting case of $\mathcal{A}^{\alpha}$ where $\alpha \to \infty$.*

More precisely, using the parameter z instead of $\alpha = \frac{z-1}{z+1}$ and writing ${}^{(z)}\mathcal{A} = \mathcal{A}^{\alpha}$, we see that

$${}^{(x+iy)}\mathcal{A} = {}^{(iy)}\mathcal{A}\ (\mathcal{B}^{\frac{1}{2}})^{x}\ .$$

Comparing this with (4.97), we conclude that

$$\mathcal{E} = {}^{(-1)}\mathcal{A} = \mathcal{A}^{\infty}.$$

Summing up, the Toeplitz, Weyl and Wick calculi correspond to the points 0, -1, and ∞ respectively in the α-plane, and the points 0, 1, and -1 respectively in the z-planes.

The polynomials q_k (4.81) have an interesting orthogonality property.

Proposition 4.3 *The polynomials $\{q_k\}_{k=1}^{\infty}$ are orthogonal with respect to the probability measure*

$$d\rho(x) := \frac{1}{(d-1)!}\chi_{(-\infty,0)}(x)\,|x|^{d-1}e^{-|x|}\,dx,$$

where $\chi_{(-\infty,0)}(x)$ is the indicator function of $(-\infty, 0)$. Moreover,

$$\|q_k\|^2_{L^2(\mathbb{R},\rho)} = \frac{(d)_k}{k!}.$$

Thus, the sequence of normalized polynomials $\{(k!/(d)_k)^{\frac{1}{2}}\,q_k(x)\}_{k=0}^{\infty}$ is an orthonormal basis of $L^2((-\infty,0),\rho)$.

Proof: Let $0 \le \ell < k$. Using the change of variables $t = -x$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} x^\ell q_k(x)\, d\rho(x) &= \frac{(-1)^\ell (d)_k}{k!(d-1)!} \int_0^\infty t^\ell \left(\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{t^j}{(d)_j} \right) t^{d-1} e^{-t}\, dt \\
&= \frac{(-1)^\ell (d)_k}{k!} \sum_{j=0}^{\ell} (-1)^j \binom{k}{j} \frac{(d)_{\ell+j}}{(d)_j} \\
&= \frac{(-1)^{\ell+d-1} (d)_k}{k!} \left(\frac{\partial}{\partial t}\right)^\ell \left(\sum_{j=0}^{k} \binom{k}{j} t^{d+j+\ell-1} \right)_{|t=-1} \\
&= \frac{(-1)^{\ell+d-1} (d)_k}{k!} \left(\frac{\partial}{\partial t}\right)^\ell \left(t^{d+\ell-1} (1+t)^k \right)_{|t=-1} = 0,
\end{aligned}$$

since $k > \ell$. Thus, $\langle f, q_k \rangle_{L^2(\rho)} = 0$ for every polynomial f of degree at most $k-1$. In particular, $\{q_k\}_{k=0}^\infty$ are orthogonal in $L^2(\rho)$. The same calculations yield

$$\int_{\mathbb{R}} x^k q_k(x)\, d\rho(x) = \frac{(-1)^{k+d-1} (d)_k}{k!} \left(\frac{\partial}{\partial t}\right)^k \left(t^{d+k-1} (1+t)^k \right)_{|t=-1} = (d)_k.$$

Thus,

$$\begin{aligned}
\int_{\mathbb{R}} q_k(x)^2\, d\rho(x) &= \sum_{\ell=0}^{k} \binom{d+k-1}{d+\ell-1} \frac{1}{\ell!} \int_{\mathbb{R}} x^\ell q_k(x)\, d\rho(x) \\
&= \frac{1}{k!} \int_{\mathbb{R}} x^k q_k(x)\, d\rho(x) = \frac{(d)_k}{k!}.
\end{aligned}$$

Q.E.D

We close this section with an example of invariant symbolic calculus which generalizes Example 4.4. It is more complicated but, nevertheless, explicitly solvable.

Example 4.5: Fix $2 \le n \in \mathbb{N}$ and $\alpha \in \mathbb{C} \setminus \{1\}$, and define

$$t_k = \begin{cases} \alpha^k & k \equiv 0 \pmod{n} \\ 0 & k \not\equiv 0 \pmod{n} \end{cases}$$

and $T = \sum_{k \ge 0} t_k\, P_k$. Let $\omega = e^{\frac{2\pi i}{n}}$ and $\alpha_\ell = \omega^\ell \alpha$. Then $\frac{1}{n} \sum_{\ell=0}^{n-1} (\alpha_\ell)^k = t_k$ for all k. Let $T^{(\ell)} = (1-\alpha_\ell)^d \sum_{k \ge 0} (\alpha_\ell)^k P_k$. Then $\frac{1}{n} \sum_{\ell=0}^{n-1} \frac{T^{(\ell)}}{(1-\alpha_\ell)^d} = T$, and hence $\frac{1}{n} \sum_{\ell=0}^{n-1} \frac{\mathcal{A}^{T^{(\ell)}}}{(1-\alpha_\ell)^d} = \mathcal{A}^T$. It follows that if $a, b \in \mathbb{C}^d$ and $\langle a|b \rangle = \lambda$ then for all $z, w \in \mathbb{C}^d$

$$\mathfrak{a}_T(\lambda) = \frac{\mathcal{A}^T_{e_{a,b}}(K_w)(z)}{K_w(z)\; e^{\langle a|w \rangle}\; e^{\langle z|b \rangle}} = \frac{1}{n} \sum_{\ell=0}^{n-1} \frac{1}{(1-\alpha_\ell)^d} \;\exp\left\{ \frac{1}{1-\alpha_\ell} \frac{\lambda}{\nu} \right\}.$$

5 Application to Toeplitz calculus and the eigenvalues of the Berezin transform on symmetric tube domains

Let $T(\Omega)$ be a symmetric tube domain of rank r and genus p in $\mathbb{C}^d$ (see subsection 1.3). Let $\nu > p-1$ and for each $b \in L^\infty(T(\Omega))$ let

$$\mathcal{T}_b = P^{(\nu)} M_b|_{L^2_a(T(\Omega),\mu_\nu)} \tag{5.98}$$

be the Toeplitz operator with symbol b on the weighted Bergman space $L^2_a(T(\Omega),\ \mu_\nu)$. Here $P^{(\nu)} : L^2(T(\Omega),\mu_\nu) \to L^2_a(T(\Omega),\mu_\nu)$ is the orthogonal projection and $M_b f = b f$ is the operator of multiplication by b. In this case the link transform is the Berezin transform $\mathcal{B}^{(\nu)} = \mathcal{T}'\mathcal{T}$. Our goal here is to give a new proof, via Theorem 2.1 of the following known result [UU94] (see also [Be78]). Here, for any $\underline{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathbb{C}^r$ we denote

$$\underline{\lambda}^* = (\lambda_r, \lambda_{r-1}, \dots, \lambda_1). \tag{5.99}$$

Also, $\underline{\rho} = (\rho_1, \rho_2, \dots, \rho_r)$, the half sum of the positive roots, is given by

$$\rho_j = (j-1)\,\frac{a}{2} + \frac{1}{2}, \qquad j = 1, 2, \dots, r, \tag{5.100}$$

and for simplicity we put also

$$t_\nu = \nu - \frac{p-1}{2}.$$

Theorem 5.1 *Let $\nu > p-1$. Then*

$$Dom(\widetilde{\mathcal{B}^{(\nu)}}) = \{\underline{\lambda} \in \mathbb{C}^r;\ |Re\,\lambda_j| < t_\nu\}$$

and for $\underline{\lambda} \in Dom(\widetilde{\mathcal{B}^{(\nu)}})$,

$$\widetilde{\mathcal{B}^{(\nu)}}(\underline{\lambda}) = \frac{\Gamma_\Omega(\underline{\lambda} + \underline{\rho} + \nu - \frac{d}{r})\,\Gamma_\Omega(-\underline{\lambda}^* - \underline{\rho}^* + \nu)}{\Gamma_\Omega(\nu - \frac{d}{r})\,\Gamma_\Omega(\nu)} = \prod_{j=1}^{r} \frac{\Gamma(t_\nu + \lambda_j)\,\Gamma(t_\nu - \lambda_j)}{\Gamma(t_\nu + \rho_j)\,\Gamma(t_\nu - \rho_j)}. \tag{5.101}$$

In the derivation of (5.101) we shall use the following result, which is of independent interest. For $\underline{\alpha}, \underline{\beta}, \underline{\gamma} \in \mathbb{R}^r$ and $z, w \in T(\Omega)$ consider the integral

$$I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}(z,w) = \int_{T(\Omega)} N_{\underline{\alpha}}(\tau(z,\xi))\ N_{\underline{\beta}}(\tau(\xi,\xi))\ N_{\underline{\gamma}}(\tau(\xi,w))\ d\mu_0(\xi). \tag{5.102}$$

Definition 5.1 *$\mathcal{D}$ is the set of all $(\underline{\alpha}, \underline{\beta}, \underline{\gamma})$ in $\mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R}^r$ such that the integral* (5.102) is absolutely convergent for all $(z,w) \in T(\Omega) \times T(\Omega)$, and the convergence is uniform on compact subsets of $T(\Omega) \times T(\Omega)$.

Theorem 5.2 *We have the inclusion*

$$\mathcal{D} \supseteq \{(\underline{\alpha}, \underline{\beta}, \underline{\gamma}) \in \mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R}^r;\ \alpha_j + \gamma_j < (j-1)\,\frac{a}{2} - p + 1,$$
$$\alpha_j + \beta_j + \gamma_j < (r-j)\,\frac{a}{2},\ \ \beta_j > \frac{d}{r} + (j-1)\,\frac{a}{2}\ \ \text{for}\ \ 1 \le j \le r\},$$

and for $(\underline{\alpha}, \underline{\beta}, \underline{\gamma}) \in \mathcal{D}$,

$$I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}(z, w) = I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}\, N_{\underline{\alpha}+\underline{\beta}+\underline{\gamma}}(\tau(z, w))$$

with

$$I_{\underline{\alpha},\underline{\beta},\underline{\gamma}} := \frac{(4\pi)^d\, \Gamma_\Omega(\underline{\beta} - \frac{d}{r})\; \Gamma_\Omega(-\underline{\alpha}^* - \underline{\beta}^* - \underline{\gamma}^*)}{\Gamma_\Omega(-\underline{\alpha}^*)\, \Gamma_\Omega(-\underline{\gamma}^*)}.$$

Recall that $X \equiv \mathbb{R}^d$ is the Euclidean Jordan algebra whose positive cone is Ω.

Lemma 5.1 *Let* $(\underline{\alpha}, \underline{\beta}, \underline{\gamma}) \in \mathcal{D}$ *and let* $g \in NA$. *Then*

$$I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}(g(z), g(w)) = N_{\underline{\alpha}+\underline{\beta}+\underline{\gamma}}(\tau(g(ie)))\, I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}(z, w)$$

for all $z, w \in T(\Omega)$.

Proof: Observe first that for all $\underline{s} \in \mathbb{C}^r$ and $g \in NA$

$$N_{\underline{s}}(\tau(g(z), g(w))) = N_{\underline{s}}(\tau(z, w))\, N_{\underline{s}}(\tau(g(ie))), \quad z, w \in T(\Omega). \tag{5.103}$$

Indeed, both sides of (5.103) are holomorphic in z, anti-holomorphic in w, and coincide for $z = w$ (see (1.27)). Therefore they coincide everywhere. Using the invariance of μ_0, we obtain

$$\begin{aligned} I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}(g(z), g(w)) &= \int_{T(\Omega)} N_{\underline{\alpha}}(\tau(g(z), \xi))\; N_{\underline{\beta}}(\tau(\xi, \xi))\; N_{\underline{\gamma}}(\tau(\xi, g(w)))\; d\mu_0(\xi) \\ &= \int_{T(\Omega)} N_{\underline{\alpha}}(\tau(g(z), g(\eta)))\; N_{\underline{\beta}}(\tau(g(\eta), g(\eta)))\; N_{\underline{\gamma}}(\tau(g(\eta), g(w)))\; d\mu_0(\eta) \\ &= N_{\underline{\alpha}+\underline{\beta}+\underline{\gamma}}(\tau(g(ie)))\, I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}(z, w). \end{aligned}$$

Q.E.D.

Corollary 5.1 *For* $(\underline{\alpha}, \underline{\beta}, \underline{\gamma}) \in \mathcal{D}$ *and all* $z, w \in T(\Omega)$,

$$I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}(z, w) = I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}(ie, ie)\; N_{\underline{\alpha}+\underline{\beta}+\underline{\gamma}}(\tau(z, w)). \tag{5.104}$$

Proof: Lemma 5.1 implies (5.104) for $z = w$ (by using $g = g_z$). Since both sides of (5.104) are holomorphic in z and anti-holomorphic in w, they coincide for all $z, w \in T(\Omega)$. Q.E.D.

Lemma 5.2 *Fix* $v \in \Omega$ *and* $\underline{\alpha} \in \mathbb{R}^r$. *Then the function*

$$f_{\underline{\alpha},v}(x) := N_{\underline{\alpha}}\left(\frac{x + iv}{2i}\right) \tag{5.105}$$

belongs to $L^2(X)$ *if and only if*

$$\alpha_j < (j-1)\,\frac{a}{4} - \frac{p-1}{2} \quad \textit{for} \quad j = 1, 2, \ldots, r. \tag{5.106}$$

Thus, if $\underline{\alpha}, \underline{\gamma} \in \mathbb{R}^r$ *satisfy*

$$\alpha_j + \gamma_j < (j-1)\,\frac{a}{2} - p + 1 \quad \textit{for} \quad j = 1, 2, \ldots, r \tag{5.107}$$

then $\int_X |\overline{f_{\underline{\alpha},v}(x)}\, f_{\underline{\gamma},v}(x)|\, dx < \infty$, and

$$\int_X \overline{f_{\underline{\alpha},v}(x)}\; f_{\underline{\gamma},v}(x)\, dx = (4\pi)^d \, \frac{\Gamma_\Omega(-\underline{\alpha}^* - \underline{\gamma}^* - \frac{d}{r})}{\Gamma_\Omega(-\underline{\alpha}^*)\,\Gamma_\Omega(-\underline{\gamma}^*)}\; N_{\underline{\alpha}+\underline{\gamma}+\frac{d}{r}}(v).$$

Proof: Notice that if $-\alpha_{r+1-j} > (j-1)\,\frac{a}{2}$ for $1 \le j \le r$ then (1.29) implies that

$$f_{\underline{\alpha},v}(x) = \frac{1}{\Gamma_\Omega(-\underline{\alpha}^*)} \int_\Omega e^{-\langle \frac{x+iv}{2i} | t\rangle} N^*_{-\underline{\alpha}^* - \frac{d}{r}}(t)\, dt$$

and the integral converges absolutely. Thus the Fourier transform satisfies

$$\hat{f}_{\underline{\alpha},v}(t) = \frac{2^{-\sum_{j=1}^{r}\alpha_j}}{\Gamma_\Omega(-\underline{\alpha}^*)}\; \chi_\Omega(t)\, e^{-\langle v|t\rangle} N^*_{-\underline{\alpha}^* - \frac{d}{r}}(t)\;.$$

Thus by Parseval's formula

$$\begin{aligned}
\|f_{\underline{\alpha},v}\|^2_{L^2(X)} &= \frac{(2\pi)^d\, 2^{-2\sum_{j=1}^{r}\alpha_j}}{\Gamma_\Omega(-\underline{\alpha}^*)^2} \int_\Omega e^{-\langle v|2t\rangle} N^*_{-2\underline{\alpha}^* - 2\frac{d}{r}}(t)\, dt \\
&= \frac{(4\pi)^d}{\Gamma_\Omega(-\underline{\alpha}^*)^2} \int_\Omega e^{-\langle v|s\rangle} N^*_{-2\underline{\alpha}^* - \frac{d}{r}}(s)\, d\mu_\Omega(s) \\
&= (4\pi)^d\, \frac{\Gamma_\Omega(-2\underline{\alpha}^* - \frac{d}{r})}{\Gamma_\Omega(-\underline{\alpha}^*)^2}\; N_{2\underline{\alpha}+\frac{d}{r}}(v)
\end{aligned}$$

provided

$$-2\alpha_{r+1-j} > \frac{d}{r} + (j-1)\,\frac{a}{2} \qquad \text{for} \quad j = 1, 2, \ldots, r,$$

which is equivalent to (5.106). Suppose now that (5.107) holds. Then

$$\int_X \overline{f_{\alpha,v}(x)}\; f_{\gamma,v}(x)\, dx = \int_X |f_{\frac{\alpha+\gamma}{2},v}(x)|^2\, dx < \infty,$$

and Parseval's theorem yields

$$\begin{aligned}
\int_X \overline{f_{\underline{\alpha},v}(x)}\; f_{\underline{\gamma},v}(x)\, dx &= \frac{2^{-\sum_{j=1}^{r}(\alpha_j+\gamma_j)}\,(2\pi)^d}{\Gamma_\Omega(-\underline{\alpha}^*)\,\Gamma_\Omega(-\underline{\gamma}^*)} \int_\Omega e^{-\langle v|2t\rangle}\; N^*_{-\underline{\alpha}^*-\underline{\gamma}^*-2\frac{d}{r}}(t)\, dt \\
&= (4\pi)^d\, \frac{\Gamma_\Omega(-\underline{\alpha}^* - \underline{\gamma}^* - \frac{d}{r})}{\Gamma_\Omega(-\underline{\alpha}^*)\,\Gamma_\Omega(-\underline{\gamma}^*)}\; N_{\underline{\alpha}+\underline{\gamma}+\frac{d}{r}}(v).
\end{aligned}$$

Q.E.D.

Proof of Theorem 5.2: In view of (5.104) we have only to find conditions for the finiteness of $I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}(ie, ie)$, and to compute it. Notice first that, with the notation (5.105), we have

$$I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}(ie, ie) = \int_\Omega \left(\int_X \overline{f_{\underline{\alpha},e+y}(x)}\; f_{\underline{\gamma},e+y}(x)\, dx \right) N_{\underline{\beta}-p}(y)\, dy.$$

Therefore, if (5.107) holds, we obtain

$$\begin{aligned} I_{\underline{\alpha},\underline{\beta},\underline{\gamma}}(ie,ie) &= \frac{(4\pi)^d\,\Gamma_\Omega(-\underline{\alpha}^*-\underline{\gamma}^*-\frac{d}{r})}{\Gamma_\Omega(-\underline{\alpha}^*)\,\Gamma_\Omega(-\underline{\gamma}^*)} \int_\Omega N_{\underline{\alpha}+\underline{\gamma}+\frac{d}{r}}(e+y)\; N_{\underline{\beta}-p}(y)\,dy \\ &= \frac{(4\pi)^d}{\Gamma_\Omega(-\underline{\alpha}^*)\,\Gamma_\Omega(-\underline{\gamma}^*)} \int_\Omega \left(\int_\Omega e^{-\langle y|t\rangle} N_{\underline{\beta}-p}(y)\,dy \right) e^{-tr(t)} N^*_{-\underline{\alpha}^*-\underline{\gamma}^*-2\frac{d}{r}}(t)\,dt \\ &= \frac{(4\pi)^d\Gamma_\Omega(\underline{\beta}-\frac{d}{r})}{\Gamma_\Omega(-\underline{\alpha}^*)\,\Gamma_\Omega(-\underline{\gamma}^*)} \int_\Omega e^{-tr(t)} N^*_{-\underline{\alpha}^*-\underline{\beta}^*-\underline{\gamma}^*-\frac{d}{r}}(t)\,dt \\ &= \frac{(4\pi)^d\Gamma_\Omega(\underline{\beta}-\frac{d}{r})\,\Gamma_\Omega(-\underline{\alpha}^*-\underline{\beta}^*-\underline{\gamma}^*)}{\Gamma_\Omega(-\underline{\alpha}^*)\,\Gamma_\Omega(-\underline{\gamma}^*)}\,, \end{aligned}$$

provided we have also

$$\beta_j > \frac{d}{r} + (j-1)\,\frac{a}{2}, \quad j = 1,2,\ldots,r$$

and

$$\alpha_j + \beta_j + \gamma_j < (r-j)\,\frac{a}{2}, \quad j = 1,2,\ldots,r.$$

This completes the proof of Theorem 5.2. Q.E.D.

For the proof of Theorem 5.1 we shall need also the following identity.

Lemma 5.3 *Let* $\underline{\alpha} \in \mathbb{C}^r$ *satisfy* $\mathrm{Re}\,(\alpha_j) > (j-1)\,\frac{a}{2}$ *for* $j = 1,2,\ldots,r$. *Then*

$$\Gamma_\Omega(\underline{\alpha}) = \Gamma_\Omega(\underline{\alpha}^* + 2\underline{\rho} - \frac{d}{r})$$

where $\underline{\rho}$ *is defined by* (5.100).

Proof: Notice first that for all $1 \le j \le r$

$$(\underline{\alpha}^* + 2\underline{\rho} - \frac{d}{r})_j - (j-1)\,\frac{a}{2} = \alpha_{r+1-j} - (r-j)\,\frac{a}{2},$$

and so the real parts of both sides are positive simultaneously. Next, when $\mathrm{Re}\,\alpha_j > (j-1)\,\frac{a}{2}$ for all j,

$$\begin{aligned} \Gamma_\Omega(\underline{\alpha}) &= (2\pi)^{\frac{d-r}{2}} \prod_{j=1}^r \Gamma(\alpha_j - (j-1)\,\frac{a}{2}) = (2\pi)^{\frac{d-r}{2}} \prod_{j=1}^r \Gamma(\alpha_{r+1-j} - (r-j)\,\frac{a}{2}) \\ &= (2\pi)^{\frac{d-r}{2}} \prod_{j=1}^r \Gamma((\underline{\alpha}^* + 2\underline{\rho} - \frac{d}{r})_j - (j-1)\,\frac{a}{2}) = \Gamma_\Omega(\underline{\alpha}^* + 2\underline{\rho} - \frac{d}{r}). \end{aligned}$$

Q.E.D.

Proof of Theorem 5.1: In the case of the Toeplitz calculus in the context of the spaces $L^2_a(T(\Omega),\mu_\nu)$ the function $F(z,w)$ is given by

$$F(z,w) = \frac{K(z,w)\,K(w,ie)}{K(z,ie)\,K(w,w)}.$$

We claim that

$$ {}_z\widetilde{F}(\underline{\lambda}) = c(\nu,\underline{\lambda})\, E_{\underline{\lambda}}(z), $$

where

$$ c(\nu,\underline{\lambda}) = \frac{\Gamma_\Omega(\underline{\lambda}+\underline{\rho}+\nu-\frac{d}{r})\,\Gamma_\Omega(-\underline{\lambda}^*-\underline{\rho}^*+\nu)}{\Gamma_\Omega(\nu)\,\Gamma_\Omega(\nu-\frac{d}{r})} $$

and

$$ E_{\underline{\lambda}}(z) = E_{e_{\underline{\lambda}}}(z,ie) = N_{\underline{\lambda}+\underline{\rho}}(\tau(z,ie)). $$

Indeed, using Theorem 5.2 and the definition (1.28)we find

$$ \begin{aligned} {}_z\widetilde{F}(\underline{\lambda}) &= a(\nu)\int_{T(\Omega)} \frac{N(\tau(z,w))^{-\nu}\,N(\tau(w,ie))^{-\nu}}{N(\tau(z,ie))^{-\nu}\,N(\tau(w,w))^{-\nu}}\; N_{\underline{\lambda}+\underline{\rho}}(\tau(w,w))\,d\mu_0(w) \\ &= a(\nu)\,N(\tau(z,ie))^{\nu}\,I_{-\nu,\underline{\lambda}+\underline{\rho}+\nu,-\nu}(z,ie) \\ &= \frac{\Gamma_\Omega(\underline{\lambda}+\underline{\rho}+\nu-\frac{d}{r})\,\Gamma_\Omega(-\underline{\lambda}^*-\underline{\rho}^*+\nu)}{\Gamma_\Omega(\nu)\,\Gamma_\Omega(\nu-\frac{d}{r})}\; N_{\underline{\lambda}+\underline{\rho}}(\tau(z,ie)). \end{aligned} $$

Using Theorem 5.2, we obtain

$$ \begin{aligned} \tilde{\mathcal{B}}^{(\nu)}(\underline{\lambda}) &= \int_{T(\Omega)} {}_z\widetilde{F}(\underline{\lambda})\,\overline{{}_z\widetilde{F}(-\underline{\bar{\lambda}})}\;|K^{(\nu)}(z,ie)|^2\,d\mu_\nu(z) \\ &= c(\nu,\underline{\lambda})\,\overline{c(\nu,-\underline{\bar{\lambda}})}\int_{T(\Omega)} N_{\underline{\lambda}+\underline{\rho}}(\tau(z,ie))\;\overline{N_{-\underline{\bar{\lambda}}+\underline{\rho}}(\tau(z,ie))}\;|K^{(\nu)}(z,ie)|^2\,d\mu_\nu(z) \\ &= a(\nu)\,c(\nu,\underline{\lambda})\,c(\nu,-\underline{\lambda})\,I_{-\underline{\lambda}+\underline{\rho}-\nu,\nu,\underline{\lambda}+\underline{\rho}-\nu}(ie,ie) \\ &= a(\nu)\,c(\nu,\underline{\lambda})\,c(\nu,-\underline{\lambda})(4\pi)^d\,\frac{\Gamma_\Omega(\nu-\frac{d}{r})\,\Gamma_\Omega(-2\underline{\rho}^*+\nu)}{\Gamma_\Omega(\underline{\lambda}^*-\underline{\rho}^*+\nu)\,\Gamma_\Omega(-\underline{\lambda}^*-\underline{\rho}^*+\nu)} \\ &= \frac{\Gamma_\Omega(\underline{\lambda}+\underline{\rho}+\nu-\frac{d}{r})\,\Gamma_\Omega(-\underline{\lambda}+\underline{\rho}+\nu-\frac{d}{r})}{\Gamma_\Omega(\nu)\,\Gamma_\Omega(\nu-\frac{d}{r})}, \end{aligned} $$

where we used the definitions of $c(\nu,\underline{\lambda})$, $c(\nu,-\underline{\lambda})$, $a(\nu)$ and Lemma 5.3 to obtain

$$ \Gamma_\Omega(-2\underline{\rho}^*+\nu) = \Gamma_\Omega(-2\underline{\rho}+\nu+2\underline{\rho}-\frac{d}{r}) = \Gamma_\Omega(\nu-\frac{d}{r}). $$

Notice also that by Lemma 5.3

$$ \Gamma_\Omega(-\underline{\lambda}+\underline{\rho}+\nu-\frac{d}{r}) = \Gamma_\Omega(-\underline{\lambda}^*+\underline{\rho}^*+\nu-\frac{d}{r}+2\underline{\rho}-\frac{d}{r}) = \Gamma_\Omega(-\underline{\lambda}^*-\underline{\rho}^*+\nu), $$

since

$$ (2\underline{\rho}^*+2\underline{\rho})_j = \frac{2d}{r} \qquad \text{for} \quad j=1,2,\ldots,r. $$

Therefore the eigenvalue of the Berezin transform $\mathcal{B}^{(\nu)}$ can be written in the form

$$ \widetilde{\mathcal{B}^{(\nu)}}(\underline{\lambda}) = \frac{\Gamma_\Omega(\underline{\lambda}+\underline{\rho}+\nu-\frac{d}{r})\,\Gamma_\Omega(-\underline{\lambda}^*-\underline{\rho}^*+\nu)}{\Gamma_\Omega(\nu)\,\Gamma_\Omega(\nu-\frac{d}{r})}. $$

Finally, using (1.26) and (5.99) we obtain

$$\begin{aligned}\widetilde{\mathcal{B}^{(\nu)}}(\underline{\lambda}) &= \prod_{j=1}^{r} \frac{\Gamma(\lambda_j+\rho_j+\nu-\frac{d}{r}-(j-1)\,\frac{a}{2})\,\Gamma(-\lambda_j+\rho_j+\nu-\frac{d}{r}-(j-1)\,\frac{a}{2})}{\Gamma(\nu-(j-1)\,\frac{a}{2})\,\Gamma(\nu-\frac{d}{r}-(j-1)\,\frac{a}{2})} \\ &= \prod_{j=1}^{r} \frac{\Gamma(\lambda_j+t_\nu)\,\Gamma(-\lambda_j+t_\nu)}{\Gamma(\rho_j+t_\nu)\,\Gamma(-\rho_j+t_\nu)}\end{aligned}$$

since $\rho_j - \frac{d}{r} - (j-1)\,\frac{a}{2} = -\frac{p-1}{2},\ \ t_\nu = \nu - \frac{p-1}{2}$ and

$$\prod_{j=1}^{r}\Gamma(\nu-(j-1)\,\frac{a}{2}) = \prod_{j=1}^{r}\Gamma(\nu-(r-j)\,\frac{a}{2}) = \prod_{j=1}^{r}(\rho_j+t_\nu).$$

This completes the proof of Theorem 5.1. Q.E.D.

Remark 5.1: Quite generally, it is easy to see that the fundamental function $\mathfrak{a}_{\mathcal{T}}(\underline{\lambda})$ of the Toeplitz calculus $\mathcal{T}$ is equal to the eigenvalue of the Berezin transform $\mathcal{B} = \mathcal{T}'\,\mathcal{T}$:

$$a_{\mathcal{T}}(\underline{\lambda}) = \widetilde{\mathcal{B}}(\underline{\lambda}).$$

Therefore Theorem 5.1 yields (3.76).

Remark 5.2: The right hand side of (5.101) is an entire meromorphic function of $\underline{\lambda}$ which is analytic in the tube

$$Q_\nu := \{\underline{\lambda} \in \mathbb{C}^r;\ \ |\mathrm{Re}\,(\lambda_\mathrm{j})| < \mathrm{t}_\nu\}. \tag{5.108}$$

Therefore the above proofs show that (5.101) holds for all $\underline{\lambda} \in Q_\nu$, $\nu > p-1$. We conjecture that it is possible to consider the Toeplitz quantization $\mathcal{T}^{(\nu)}$ and the associated link transform $\mathcal{B}^{(\nu)}$ in a canonical and explicit way for all $\nu > \frac{p-1}{2}$, and that (5.101) is valid for all $\underline{\lambda} \in Q_\nu$ in the extended range of ν. This may require the techniques of [AU97] and [AU99].

References

[A96] J. Arazy *Boundedness and compactness of generalized Hankel operators on bounded symmetric domains*, J. Funct. Anal., **137** (1996), 97-151.

[AU97] J. Arazy and H. Upmeier, *Invariant Inner Products in Spaces of Holomorphic Functions on Bounded Symmetric Domains*, Documenta Mathematica **2,** 213-261 (1997).

[AU98] J. Arazy and H. Upmeier, *Discrete Series Representations and Integration over Boundary Orbits of Symmetric Domains*, Perspectives on Quantization (Eds. L. A. Coburn and M. A. Rieffel) Contemp. Math. Vol. **214,** 1-22 (1998).

[AU99] J. Arazy and H. Upmeier, *Boundary integration and the discrete Wallach points*, ESI preprint No. **762**, (1999), to be submitted to AMS Memoirs.

[AU01] J. Arazy and H. Upmeier, *Invariant symbolic calculi and eigenvalues of invariant operators on real symmetric domains*, in preparation.

[Be71] F. Berezin, *Wick and anti-Wick operator symbols*, Mat. Sb. **86** (1971), No. 4, 578-610.

[Be72] F. Berezin, *Covariant and contravariant symbols of operators*, Math. USSR-Izv. **6** (1972) No. 5, 1117-1151.

[Be73] F. Berezin, *Quantization in complex bounded domains*, Soviet Math. Dokl. **14** (1973) No. 4, 1209-1213.,

[Be74-1] F. Berezin, *General concept of Quantization*, Comm. Math. Phys. **40** (1975), 153-174.

[Be74-2] F. Berezin, *Quantization*, Math. USSR-Izv. **8** (1974) No. 5, 1109-1165.

[Be75] F. Berezin, *Quantization in complex symmetric spaces*, Math. USSR-Izv. **9** (1975) No. 2, 341-379.

[Be78] F. Berezin, *On relation between covariant and contravariant symbols of operators for complex classical domains*, Soviet Math. Dokl.**19** (1978), 786-789.

[E98] M. Englis, *Invariant operators and the Berezin transform on Cartan domains*, Math. Nachr. **195** (1998), 61-75.

[FK90] J. Faraut and A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), 64-89.

[FK94] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford (1994).

[Gi64] S. Gindikin, *Analysis on homogeneous domains*, Russ. Math. Surv. **19** (1964), 1-89.

[Gi75] S. Gindikin, *Invariant generalized functions in homogeneous domains*, Funct. Anal. Appl., **9** (1975), 50-52.

[He78] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric spaces*, Academic Press (1978).

[He84] S. Helgason, *Groups and geometric analysis*, Academic Press (1984).

[Hu63] L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, Transl. Math. Monographs, Amer. Math. Soc. (1963).

[La86] M. Lassalle, *Noyau de Szegö, K-types et algèbres de Jordan*, C. R. Acad. Sci. Paris **303** (1986), 1-4.

[La87] M. Lassalle, *Algèbres de Jordan et ensemble de Wallach*, Invent. Math. **8**9 (1987), 375-393.

[Lo77] O. Loos, *Bounded Symmetric Domains and Jordan Pairs*, Univ. California, Irvine (1977).

[Sch69] W. Schmid, *Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen*, Invent. Math. **8** (1969), 61-80.

[Un98] A. Unterberger, *Quantization, Symmetries and Relativity*, Contemporary Mathematics vol. **214** (1998), 169 - 187.

[Up87] H. Upmeier, *Jordan algebras in analysis, operator theory, and quantum mechanics*, CBMS Series in Math. 67, Amer. Math. Soc., (1987).

[UU94] A. Unterberger and H. Upmeier, *Berezin Transform and Invariant Differential Operators*, Comm. Math. Phys., **164** (1994), 563-597.

[UU96] A. Unterberger and H. Upmeier, *Pseudodifferential Analysis on symmetric cones*, CRC Press, Boca Raton (1996)

[VR76] M. Vergne and H. Rossi, *Analytic continuation of holomorphic discrete series of a semi-simple Lie group*, Acta Math. **136** (1975), 1-59.

[W79] N. Wallach, *The analytic continuation of the discrete series*, I, II, Trans. Amer. Math. Soc. **251** (1979), 1-17, and 19-37.