

THE BETTI POLYNOMIALS OF POWERS OF AN IDEAL

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ABSTRACT. For an ideal I in a regular local ring or a graded ideal I in the polynomial ring we study the limiting behavior of $\beta_i(S/I^k) = \dim_K \operatorname{Tor}_i^S(S/\mathfrak{m}, S/I^k)$ as k goes to infinity. By Kodiyalam's result it is known that $\beta_i(S/I^k)$ is a polynomial for large k . We call these polynomials the Kodiyalam polynomials and encode the limiting behavior in their generating polynomial. It is shown that the limiting behavior depends only on the coefficients on the Kodiyalam polynomials in the highest possible degree. For these we exhibit lower bounds in special cases and conjecture that the bounds are valid in general. We also show that the Kodiyalam polynomials have weakly descending degrees and identify a situation where the polynomials have all highest possible degree.

1. INTRODUCTION

Let S be either a regular local ring with maximal ideal \mathfrak{m} and residue class field K or a polynomial ring over K with maximal graded ideal \mathfrak{m} . We assume that $\dim S = n$. Furthermore, let I be a proper (graded) ideal in S . In his paper [7] Kodiyalam proved that

$$\beta_i(S/I^k) = \dim_K \operatorname{Tor}_i^S(S/\mathfrak{m}, S/I^k)$$

as a function of k is a polynomial function of degree $\leq \ell(I) - 1$ for $k \gg 0$. Here $\ell(I)$ denotes the *analytic spread* of I , that is, the Krull-dimension of the fiber $R(I)/\mathfrak{m}R(I)$ of the Rees algebra $R(I) = \bigoplus_{k \geq 0} I^k t^k$. It is known and easy to prove that $\operatorname{height}(I) \leq \ell(I) \leq \dim S$.

We denote by $\mathfrak{P}_i(I)$ the polynomial with $\mathfrak{P}_i(I)(k) = \beta_i(S/I^k)$ for $k \gg 0$. and call the polynomials $\mathfrak{P}_0(I), \mathfrak{P}_1(I), \dots, \mathfrak{P}_n(I)$ the *Kodiyalam polynomials* of I . Note that $\mathfrak{P}_0(I) = 1$.

It is an immediate consequence of Kodiyalam's result, see Remark 2.1, that the projective dimension $\operatorname{pd}(S/I^k)$ of S/I^k stabilizes for $k \gg 0$. Indeed this fact was proved by different means first by Brodmann [3]. Note, that Brodmann's result was formulated in terms of the depth rather than the projective dimension. We write $\operatorname{apd}(I)$ for $\lim_{k \rightarrow \infty} \operatorname{pd}(S/I^k)$ and call $\operatorname{apd}(I)$ the *asymptotic projective dimension* of I .

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In this paper we are interested in the limiting behavior of the polynomial

$$\mathfrak{P}(I)(k, t) = \sum_{i=0}^{\text{apd}(I)} \mathfrak{P}_i(I)(k) t^{\text{apd}(I)-i}$$

as k goes to infinity. Clearly, at least $\mathfrak{P}_1(I)(k)$ goes to infinity if $\ell(I) \neq 1$. Indeed, in Proposition 2.2 we show that $\ell(I) - 1 = \deg \mathfrak{P}_1(I) \geq \deg \mathfrak{P}_2(I) \geq \dots \geq \deg \mathfrak{P}_{\text{apd}(I)}(I)$. In the proof of Proposition 2.2, essentially following the ideas by Kodiyalam [7], we identify $\mathfrak{P}_i(I)$ as the Hilbert polynomial of the some finitely generated module. Therefore, the leading coefficient of $\mathfrak{P}_i(I)$ is of the form $k_i/d_i!$ where $d_i = \deg \mathfrak{P}_i(I)$. By $\mathfrak{K}(I)$ we denote $\max\{i \mid d_i = \ell(I) - 1\}$. Note, that the preceding facts imply that k_i is the multiplicity of a finitely generated module.

We show that the limiting behavior for $k \rightarrow \infty$ of $\mathfrak{P}(I)(k, t)$ is up to convergence rate completely determined by the polynomial $\sum_{i=1}^{\mathfrak{K}(I)} k_i \cdot t^{\text{apd}(I)-i}$. More precisely:

Theorem 1.1. *Let I be a (graded) ideal in S such that $\ell(I) \geq 2$. Let $\alpha_1, \dots, \alpha_{\mathfrak{K}(I)-1}$ be the roots of the polynomial $\sum_{i=1}^{\mathfrak{K}(I)} k_i \cdot t^{\text{apd}(I)-i}$. Then for $1 \leq i \leq \text{apd}(I)$ there are sequences $(\gamma_k^{(i)})_{k \geq 1}$ of complex numbers, such that after suitable numbering:*

- (i) $\prod_{i=1}^{\text{apd}(I)} (t - \gamma_k^{(i)}) = \mathfrak{P}(I)(k, t)$ for all $k \geq 1$.
- (ii) $\gamma_k^{(i)} \rightarrow \alpha_i$, $1 \leq i \leq \text{apd}(I) - 1$, for $k \rightarrow \infty$.
- (iii) $\gamma_k^{(\text{apd}-1)} = \alpha_{\text{apd}-1} = -1$, for all $k \geq 1$.
- (iv) $\gamma_k^{(\text{apd}(I))} \in \mathbb{R}$ for $k \gg 0$ and $\gamma_k^{(\text{apd}(I))} \rightarrow -\infty$ for $k \rightarrow \infty$.

The assumption $\ell(I) \geq 2$ is equivalent to saying that I is not a principal ideal. Clearly, for principal ideals I , each power I^k is principal and $\beta_0 = \beta_1 = 1$, $\beta_i = 0$ for $i \geq 2$ which is a trivial situation for our purposes.

Theorem 1.1 focuses our interest on the number $\mathfrak{K}(I)$ and the multiplicities k_i for $1 \leq i \leq \mathfrak{K}(I)$. Note, that in Theorem 1.1 the number of α_i equal to 0 is $\text{apd}(I) - \mathfrak{K}(I)$ and that for $1 \leq i \leq \mathfrak{K}(I)$ we have

$$\frac{k_i}{k_1} = \lim_{k \rightarrow \infty} \frac{\beta_i(S/I^k)}{\beta_1(S/I^k)}.$$

The following two are our main results.

Theorem 1.2. *Suppose that $\ell(I) = n$. Then $\mathfrak{K}(I) = n$, in particular, $\deg \mathfrak{P}_i(I) = n - 1$ for $i = 1, \dots, n$.*

Theorem 1.3. *Suppose that $R(I)/\mathfrak{m}R(I)$ is a domain and $R(I)_{\mathfrak{m}R(I)}$ is Cohen-Macaulay. Then $k_i/k_1 \geq \binom{\mathfrak{K}(I)-1}{i-1}$ for $i = 1, \dots, \mathfrak{K}(I)$. Moreover, equality holds if and only if $R(I)_{\mathfrak{m}R(I)}$ is a complete intersection.*

As a first corollary we get that the inequality from Theorem 1.3 holds if $\ell(I) = n$ and $R(I)/\mathfrak{m}R(I)$ is a domain. Observe that $R(I)/\mathfrak{m}R(I)$ is always a domain if I is a graded ideal in the polynomial ring generated by elements of the same degree. From this remark and Theorem 1.3 we deduce in a second corollary that equality holds for Artinian monomial ideals generated in a single degree with linear relations.

Based on experimental data we conjecture that the inequality from Theorem 1.3 holds in general.

Conjecture 1.4. *Let $I \subset S$ be a (graded) ideal. Then*

$$\lim_{k \rightarrow \infty} \frac{\beta_i(S/I^k)}{\beta_1(S/I^k)} = \frac{k_i}{k_1} \geq \binom{\mathfrak{R}(I) - 1}{i - 1} \quad \text{for } i = 1, \dots, \mathfrak{R}(I).$$

We note that the condition $\frac{k_i}{k_1} \geq \binom{\mathfrak{R}(I) - 1}{i - 1}$ from Conjecture 1.4 is satisfied whenever the polynomial $\sum_{i=1}^{\mathfrak{R}(I)} k_i \cdot t^{\text{apd}(I) - i}$ has only real roots (see [1, Observation 3.4]). Indeed, we know of no example for which the polynomial is not real rooted. But we consider our evidence too weak for a conjecture. Indeed, we see in Remark 2.5 that for $\ell(I) \geq 2$ we have that -1 is always a root. In addition, in all example we tried experimentally $\ell(I)$ was small and there were only very few roots other than -1 .

For the class of monomial ideals it is an interesting question which of the invariants defined for I in the introduction can depend on the characteristic of the field. The fact that $\ell(I)$ is independent of the field is an immediate consequence of a convex geometric description in [5] (see also [9, Corollary 4.10]). On the other hand for the invariants $\text{apd}(I)$, $\mathfrak{P}_i(I)$ for some $i > 1$, $\mathfrak{R}(I)$ and then k_i for some $i > 1$ we do not know of a proof nor a counterexample. In general counterexamples are hard to find, due to the fact that only small powers of monomial ideals can be treated with the existing computer algebra systems.

2. THE KODIYALAM POLYNOMIALS OF AN IDEAL

Before we come to a more subtle analysis of the polynomials $\mathfrak{P}_i(I)(k)$ we state a simple consequence of the fact that $\beta_i(S/I^k) = \mathfrak{P}_i(I)(k)$ for $k \gg 0$. As mentioned in the introduction the conclusion was first shown by Brodmann [3] in terms of depth.

Remark 2.1. *The projective dimension $\text{pd}(S/I^k)$ stabilizes for $k \gg 0$.*

Proof. Let $q = \max\{i: \mathfrak{P}_i(I) \neq 0\}$, and let k_0 be an integer such that $\mathfrak{P}_i(I)(k) = \beta_i(S/I^k)$ for all $k \geq k_0$. Since $\mathfrak{P}_i(I)(k)$ has only finitely many zeroes, we may also assume that $\mathfrak{P}_q(I)(k) \neq 0$ for all $k \geq k_0$. Then $\text{pd}(S/I^k) = q$ for all $k \geq k_0$. \square

For a polynomial P we set $\deg P = -\infty$ if P is the zero polynomial. Using this convention we get.

Proposition 2.2. $\ell(I) - 1 = \deg \mathfrak{P}_1(I) \geq \deg \mathfrak{P}_2(I) \geq \dots \geq \deg \mathfrak{P}_n(I)$.

Proof. For $i \geq 1$ we have

$$\beta_i(S/I^k) = \beta_{i-1}(I^k) = \dim_K \operatorname{Tor}_{i-1}^S(S/\mathfrak{m}, I^k) = \dim_K H_{i-1}(\mathbf{x}; I^k).$$

Here $H_i(\mathbf{x}; I^k)$ is the i th Koszul homology of I^k with respect to $\mathbf{x} = x_1, \dots, x_n$, where \mathbf{x} is a regular system of parameters if S is a regular local ring, and is the sequence of indeterminates in case S is a polynomial ring.

Observe that $H_i(\mathbf{x}; R(I))$ is a graded $H_0(\mathbf{x}; R(I))$ -module. Thus by $H_0(\mathbf{x}; R(I)) = R(I)/\mathfrak{m}R(I)$ it is a graded $R(I)/\mathfrak{m}R(I)$ -module. Since $H_i(\mathbf{x}; R(I))_k = H_i(\mathbf{x}; I^k)$ for all k , we see that $\mathfrak{P}_i(I)$ is the Hilbert polynomial of $H_{i-1}(\mathbf{x}; R(I))$ for $i \geq 1$. Thus the degree of $\mathfrak{P}_i(I)$ is the Krull dimension of $H_{i-1}(\mathbf{x}; R(I))$ minus 1. In particular, $\deg \mathfrak{P}_1(I) = \dim R(I)/\mathfrak{m}R(I) - 1 = \ell(I) - 1$.

In order to prove the inequalities $\deg \mathfrak{P}_{i+1}(I) \leq \deg \mathfrak{P}_i(I)$, it remains to show that $\dim H_i(\mathbf{x}; R(I)) \leq \dim H_{i-1}(\mathbf{x}; R(I))$ for all $i \geq 1$. To see this, let $P \in \operatorname{Supp} H_i(\mathbf{x}; R(I))$. Then $\mathfrak{m}R(I) \subset P$ and $H_i(\mathbf{x}; R(I)_P) = H_i(\mathbf{x}; R(I))_P \neq 0$. Rigidity of the Koszul homology (see [4, Exercise 1.6.31]) implies that $H_{i-1}(\mathbf{x}; R(I))_P = H_{i-1}(\mathbf{x}; R(I)_P) \neq 0$. Thus $\operatorname{Supp}(H_i(\mathbf{x}; R(I))) \subset \operatorname{Supp} H_{i-1}(\mathbf{x}; R(I))$, which yields the desired inequality for the dimensions. \square

We give a first example which shows that there are cases where the inequalities in Proposition 2.2 are indeed equalities.

Example 2.3. Let $I = (x^3, x^2 - yz, y^4 + xz^3, xy - z^2) \subset S = K[x, y, z]$. The ideal I is (x, y, z) -primary, so that $\ell(I) = 3$ and $\operatorname{pd} S/I^k = 3$ for all k . It follows from Theorem 1.2 that $\deg \mathfrak{P}_i(I) = 2$ for $i = 1, 2, 3$. A calculation with CoCoA indicates that $\mathfrak{P}_1(I)(k) = (k+1)^2$, $\mathfrak{P}_2(I)(t) = (\frac{5}{2}k + \frac{7}{2})k$ and $\mathfrak{P}_3(I)(k) = \frac{3}{2}k(k+1)$. So here we have $2 = \ell(I) - 1 = \deg \mathfrak{P}_1(I) = \deg \mathfrak{P}_2(I) = \deg \mathfrak{P}_3(I)$. More precisely, $k_1 = 6$, $k_2 = 15$ and $k_3 = 21$.

The second example shows that even for monomial ideals the inequalities from Proposition 2.2 can be strict.

Example 2.4. Consider the monomial ideal

$$I = (a^6, a^5b, ab^5, b^6, a^4b^4c, a^4b^4d, a^4e^2f^3)$$

in $\mathbb{Q}[a, b, c, d, e, f]$. Then $\mathfrak{P}_1(I)(k) = 3k^2 + 4k - 7$, $\mathfrak{P}_2(I)(k) = 6k^2 + 3k - 7$, $\mathfrak{P}_3(I)(k) = 3k^2 - k + 5$, $\mathfrak{P}_4(I)(k) = 5$, $\mathfrak{P}_5(I)(k) = 1$ and $\mathfrak{P}_6(I)(k) = 0$. Thus $\deg \mathfrak{P}_i(I) = 2$ for $i = 1, 2, 3$, while $\mathfrak{P}_4(I)$ and $\mathfrak{P}_5(I)$ are of degree 0, and $\mathfrak{P}_6(I)$ is the zero polynomial. In particular, $\mathfrak{R}(I) = 3$.

In the light of Proposition 2.2 and Examples 2.3 and 2.4, Theorem 1.2 provides sufficient conditions for extremal behavior of $\mathfrak{R}(I)$.

Proof of Theorem 1.2. It has been shown by Brodmann [3] that $\operatorname{pd} S/I^k \geq \ell(I)$ for $k \gg 0$. Thus our assumptions imply that $\operatorname{pd} S/I^k = n$ for $k \gg 0$. Therefore, $\mathfrak{P}_n(I) \neq 0$ and $\deg \mathfrak{P}_n(I) \geq 0$.

We will show that $\deg \mathfrak{P}_n(I) = n - 1$, equivalently, that $\dim H_{n-1}(\mathbf{x}; R(I)) = n$. Then the assertion of the theorem follows from Proposition 2.2.

Notice that

$$H_{n-1}(\mathbf{x}; R(I))_k = H_{n-1}(\mathbf{x}; I^k) \cong H_n(\mathbf{x}; S/I^k) \cong (I^k :_S \mathfrak{m})/I^k = 0 :_G \mathfrak{m}G,$$

where $G = \bigoplus_{k \geq 0} I^k/I^{k+1}$.

From here on the proof follows an argument given to us by Shiro Goto and the referee. Our assumption on the analytic spread implies that $\dim G = \dim G/\mathfrak{m}G = n$. Hence there exists a prime ideal $P \in \text{Spec } G$ with $\mathfrak{m}G \subset P$ and $\dim G/P = n$. Now in the Artinian local ring G_P we have

$$0 \neq (0 :_{G_P} PG_P) \subset (0 :_{G_P} \mathfrak{m}G_P) \cong (0 :_G \mathfrak{m}G)_P.$$

From this it follows that $P \in \text{Ass}_G(0 :_G \mathfrak{m}G)$. Consequently, $\dim(0 :_G \mathfrak{m}G) \geq n$.

□

In case I is \mathfrak{m} -primary the consequence of Theorem 1.2 was first proved using different means in [6].

We now turn our attention to the multiplicities k_i for $1 \leq i \leq \mathfrak{R}(I)$.

Remark 2.5. *If $\ell(I) \geq 2$, then*

$$\sum_{i=1}^{\mathfrak{R}(I)} (-1)^i k_i = 0.$$

Proof. Since $\sum_{i=0}^n (-1)^i \beta_i(S/I^k) = 0$ for all $k \geq 1$, it follows that $\sum_{i=0}^n (-1)^{i+1} \mathfrak{P}_i(I)(k) =$

0. All terms in the alternating sum are polynomials for $k \gg 0$. Therefore, for any k -power the alternating sum of the coefficients cancels. Now by $\ell(I) \geq 2$, the maximal degree $\ell(I) - 1 > 0 = \deg \mathfrak{P}_0(I)(k)$ is achieved for $\mathfrak{P}_i(I)(k)$, $1 \leq i \leq \mathfrak{R}(I)$. This implies the assertion. □

If one looks at the actual values of the k_i in Example 2.3 one observes that $\mathfrak{R}(I) = 2$ and $k_i/k_1 \geq \binom{2}{i-1}$, and in Example 2.4 we have $\mathfrak{R}(I) = 3$ and $k_i/k_1 = \binom{3}{i-1}$. Theorem 1.3 provides conditions under which inequalities of that type hold. Before we can proceed to the proof of Theorem 1.3 we need the following lemma.

Lemma 2.6. *Let P be a prime ideal of height h in a regular local ring (R, \mathfrak{n}, K) . Then*

$$(1) \quad \dim_K \text{Tor}_i^R(K, R/P) \geq \binom{h}{i} \quad \text{for } i = 1, \dots, h.$$

Equality holds if and only if P is generated by a regular sequence.

Proof. Let \mathbb{F} be a minimal free R -resolution of R/P . The ring R_P is a regular local ring of dimension h , and the localization \mathbb{F}_P is a free resolution of the residue class field R_P/PR_P . Since PR_P is generated by a regular sequence of length h , we see that

$$\dim_K \operatorname{Tor}_i^R(K, R/P) = \operatorname{rank}_R F_i = \operatorname{rank}_{R_P} (F_i)_P \geq \binom{h}{i-1}.$$

On the other hand, if P is generated by a regular sequence, then the Koszul complex of this sequence provides a minimal free R -resolution of R/P , and equality holds in (1).

Conversely, suppose we have equality in (1). Then $\dim_K \operatorname{Tor}_1^R(K, R/P) = h$, which implies that P is generated by h elements. Since h is the height of P , these elements form a regular sequence \square

Proof of Theorem 1.3. The proof of Theorem 1.1 implies that the multiplicity of the $R(I)/\mathfrak{m}R(I)$ -module $H_{i-1}(\mathbf{x}; R(I))$ is k_i . In particular, k_1 is the multiplicity of $R(I)/\mathfrak{m}R(I) = H_0(\mathbf{x}; R(I))$. Hence by [4, Corollary 4.6.9] it follows that

$$k_i = k_1 \cdot \operatorname{rank} H_{i-1}(\mathbf{x}; R(I)) \quad \text{for } i = 1, \dots, \mathfrak{K}(I).$$

Set $T = R(I)_{\mathfrak{m}R(I)}$ and denote by W the residue class field of the local ring T . Then for $i = 1, \dots, \mathfrak{K}(I)$ the rank of $H_{i-1}(\mathbf{x}; R(I))$ is the vector space dimension of the W -vector space $H_{i-1}(\mathbf{x}; T)$. Since \mathbf{x} is a system of generators of $\mathfrak{m}R(I)$, the numbers $\dim_W H_{i-1}(\mathbf{x}; T)$ have the following interpretation: suppose I is generated by f_1, \dots, f_m . Let $A = S[y_1, \dots, y_m]$ be the polynomial over S in the variables y_i . Let J denote the kernel of the canonical, surjective S -algebra homomorphism $\varphi: A \rightarrow R(I)$ with $y_i \mapsto f_i$ for $i = 1, \dots, m$, and set $P = (J, \mathfrak{m})$. Then P is a prime ideal and $B = A_P$ is a regular local ring. The algebra homomorphism φ induces then a surjective homomorphism $B \rightarrow T$ of local rings, and it follows that

$$\dim_W H_{i-1}(\mathbf{x}; T) = \dim_W \operatorname{Tor}_{i-1}^B(W, T) \quad \text{for } i = 1, \dots, \mathfrak{K}(I).$$

In particular, $\operatorname{pd}_B T = \mathfrak{K}(I) - 1$, since $H_{\mathfrak{K}(I)-1}(\mathbf{x}; T) \neq 0$, but $H_{i-1}(\mathbf{x}; T) = 0$ for $i > \mathfrak{K}(I)$. Let H be the kernel of $B \rightarrow T$. Then H is a prime ideal with

$$\operatorname{height} H = \dim B - \dim T = \dim B - \operatorname{depth} T = \operatorname{pd}_B T = \mathfrak{K}(I) - 1.$$

Here we have employed the assumption that T is Cohen–Macaulay.

The assertions of the theorem now follow from Lemma 2.6 applied to the prime ideal H and the regular local ring B . \square

Corollary 2.7. *Suppose that $R(I)/\mathfrak{m}R(I)$ is a domain and that $\ell(I) = n$. Then*

$$\lim_{k \rightarrow \infty} \frac{\beta_i(S/I^k)}{\beta_1(S/I^k)} = \frac{k_i}{k_1} \geq \binom{n-1}{i-1} \quad \text{for } i = 1, \dots, n.$$

Proof. Since $\ell(I) = n$, it follows that $P = \mathfrak{m}R(I)$ is a prime ideal of height 1. Therefore, $R(I)_P$ is a one dimensional local domain and hence Cohen–Macaulay. Thus we may apply Theorem 1.3 and obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\beta_i(S/I^k)}{\beta_1(S/I^k)} &= \lim_{k \rightarrow \infty} \frac{\mathfrak{P}_i(I)(k)}{\mathfrak{P}_1(I)(k)} \\ &= \lim_{k \rightarrow \infty} \frac{\frac{k_i}{(n-1)!}k^{n-1} + \dots}{\frac{k_1}{(n-1)!}k^{n-1} + \dots} \\ &= \frac{k_i}{k_1}. \end{aligned}$$

□

In the next result we describe a situation in which the hypotheses of Theorem 1.3 for the equality conclusion are satisfied.

Corollary 2.8. *Let $I \subset S$ be a monomial ideal generated in a single degree with $\dim S/I = 0$. Suppose that I has linear relations. Then*

$$\lim_{k \rightarrow \infty} \frac{\beta_i(S/I^k)}{\beta_1(S/I^k)} = \frac{k_i}{k_1} = \binom{n-1}{i-1} \quad \text{for } i = 1, \dots, n.$$

Proof. Let $I = (u_1, \dots, u_m)$ be the monomial generators of I , each of degree d . Since they are all of same degree, it follows that $R(I)/\mathfrak{m}R(I) \cong K[u_1, \dots, u_m]$. In particular, $R(I)/\mathfrak{m}R(I)$ is a domain. We denote the prime ideal $\mathfrak{m}R(I)$ by P , and show that $R(I)_P$ is a discrete valuation ring. Then it follows that height $P = 1$, so that $\ell(I) = n$, and Theorem 1.3 yields the desired equations.

In order to prove that $R(I)_P$ is a discrete valuation ring, it suffices to show that $PR(I)_P$ is generated by one element. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a regular system of parameters in case S is a regular local ring and the sequence of indeterminates in case S is a polynomial ring. Observe, that $(x_1, \dots, x_n)R(I)_P = PR(I)_P$. We will show that each x_i differs from x_1 only a by unit, from which the desired conclusion will follow.

Since $\dim S/I = 0$, we have that $x_i^d \in I$ for $i = 1, \dots, n$. Let F be the free S -module with basis e_1, \dots, e_m and let $\varepsilon: F \rightarrow I$ the S -module epimorphism with $\varepsilon(e_i) = u_i$ for $i = 1, \dots, m$. Let i be an integer with $1 < i \leq m$. Since I has linear relations, the relation $x_i^d e_1 = x_1^d e_i$ can be expressed as a multihomogeneous linear combination of linear relations, namely

$$x_i^d e_1 - x_1^d e_i = \sum_j v_j r_j,$$

with v_j monomials and relations $r_j = x_{j_k} e_{j_k} - x_{j_l} e_{j_l}$, and where the multidegree of each summand is equal to the multidegree $x_1^d x_i^d$. It follows that $\{x_{j_k}, x_{j_l}\} = \{x_1, x_i\}$ for all j . We choose one of the relations r_j in this sum, and may assume that $r_j = x_1 e_{j_k} - x_i e_{j_l}$. This relation gives rise to the equation $x_1(u_{j_k} t) = x_i(u_{j_l} t)$ in the

Rees algebra $R(I)$. Since the elements $u_i t$ do not belong to P , they become units in $R(I)_P$. Thus the preceding equation shows that x_1 and x_i only differ by a unit in $R(I)_P$, as desired. \square

We note that the conclusion of Corollary 2.8 is valid in many cases that do not satisfy its assumptions.

Example 2.9. Let $I = (xy, vw, xz)$ then $\ell(I) = 3 = \text{apd}(I)$ and $\mathfrak{P}_1(I)(k) = \frac{1}{2}k^2 + \frac{3}{2} + 1$, $\mathfrak{P}_2(I)(k) = k^2 + 2k$ and $\mathfrak{P}_3(I)(k) = \frac{1}{2}k^2 + \frac{1}{2}k$. Thus $\mathfrak{R}(I) = 3$ and $k_1 = 1 = \binom{\mathfrak{R}(I)-1}{0}$, $k_2 = 2 = \binom{\mathfrak{R}(I)-1}{2}$ and $k_3 = 1 = \binom{\mathfrak{R}(I)-1}{2}$. But I does not have linear relations by $\beta_{2,4}(S/I) = 1$.

3. ROOTS OF POLYNOMIALS

Before we can prove Theorem 1.1 we need a technical lemma. A similar lemma, albeit for polynomials with a different structure, appears in [2] in another context.

Lemma 3.1. *Let $(f_k(t))_{k \geq 1}$ be a sequence of real polynomials of degree $\leq q - 1$ and $f(t)$ a non-zero real polynomial of degree $q - 1$. Assume that all $(f_k(t))_{k \geq 1}$ and $f(t)$ have non-negative coefficients. Let ℓ be a natural number such that:*

$$\triangleright \lim_{k \rightarrow \infty} f_k(t)/k^\ell = 0, \text{ where the limit is taken in } \mathbb{R}^q.$$

Let $\alpha_1, \dots, \alpha_{q-1}$ be the roots of $f(t)$. Then there are sequences $(\gamma_i^{(k)})_{k \geq 1}$, $1 \leq i \leq q$ of complex numbers such that:

- (i) $\prod_{i=1}^q (t - \gamma_i^{(k)}) = f_k(t) + k^\ell f(t) + t^q$.
- (ii) $\gamma_i^{(k)} \rightarrow \alpha_i$, $1 \leq i \leq q - 1$, for $k \rightarrow \infty$.
- (iii) $\gamma_q^{(k)}$ is real for $k \gg 0$ and $\gamma_q^{(k)} \rightarrow -\infty$ for $k \rightarrow \infty$.

Proof. Assume first that the roots α_i are pairwise disjoint. Now we consider a fixed zero α_i . Let $\varepsilon > 0$ be such that $f(t) \neq 0$ for $0 < |t - \alpha_i| < 2\varepsilon$. Set $G_\varepsilon^i = \{t \mid |t - \alpha_i| \leq \varepsilon\}$. We claim that for large enough k the polynomial $f_k(t) + k^\ell f(t) + t^q$ has a zero in G_ε^i . Assume not. Then we can find arbitrarily large k for which $g_k(t) := f_k(t) + k^\ell f(t) + t^q$ does not vanish in G_ε^i . Then $1/g_k(t)$ is holomorphic inside G_ε^i . By the maximum principle the maximum of $1/g_k(t)$ on G_ε^i is obtained on the boundary of G_ε^i . In particular, this implies that there is a t_0 such that $|t_0 - \alpha_i| = \varepsilon$ and $|1/g_k(t_0)| > |1/g_k(\alpha_i)|$. Hence $|g_k(\alpha_i)| > |g_k(t_0)|$. Thus

$$|f_k(\alpha_i) + \alpha_i^q| > |f_k(t_0) + k^\ell f(t_0) + t_0^q|$$

This implies

$$1/k^\ell |f_k(\alpha_i)| + 1/k^\ell |\alpha_i|^q > |1/k^\ell f_k(t_0) + f(t_0) + 1/k^\ell t_0^q|$$

Since by assumption the left hand side converges to 0 for $k \rightarrow \infty$ and the right hand side to $|f(t_0)| > 0$ we obtain a contradiction. Hence there is a zero of $f_k(t) + k^\ell f(t) + t^q$ in G_ε^i for large k .

Now we choose ε small enough so that the G_ε^i , $1 \leq i \leq q-1$, are pairwise disjoint. In this situation and for large enough k we denote by $\gamma_i^{(k)}$ the zero of $f_k(t) + k^\ell f(t) + t^q$ in the disk G_ε^i around α_i with radius ε . Then as ε goes to 0 the root $\gamma_i^{(k)}$ converges to α_i , $1 \leq i \leq q-1$. Since for $k \rightarrow \infty$ at least one coefficient of $g_k(t)$ goes to infinity there must be at least one root with modulus going to infinity. We call this root $\gamma_q^{(k)}$.

The argumentation so far shows that for each distinct root of $f(t)$ there is a sequence of roots of $f_k(t) + k^\ell f(t) + t^q$ converging to the root and in addition there is a sequence $\gamma_q^{(k)}$ of roots with $\lim_{k \rightarrow \infty} |\gamma_q^{(k)}| = \infty$. We are left with studying multiple roots. Assume α is an r -fold root of $f(t)$ for some $r \geq 2$. In this case α is also a root of $k^\ell \frac{\partial^i}{\partial t^i} f(t)$ for $0 \leq i \leq r-1$. Consider the polynomial

$$\frac{1}{q} \left(k^\ell \frac{\partial}{\partial t} f(t) + \frac{\partial}{\partial t} f_k(t) + q(q-1) \cdots (q-i)t^{q-1} \right).$$

By induction on r we obtain that this polynomial has $r-1$ roots converging to α as k goes to infinity. Now, by [8, Theorem 3.2.4] it follows again that there are sequences $(\gamma_i^{(k)})_{k \geq 1}$, $1 \leq i \leq q$ such that statement (i) and (ii) hold and such that $\lim_{k \rightarrow \infty} |\gamma_q^{(k)}| = \infty$. It thus remains to verify (iii).

Since by assumption at least one of the coefficients of $f_k(t) + k^\ell f(t) + t^q$ is unbounded and there are $q-1$ bounded roots it follows that there must be a q -th root that is unbounded. Since $k^\ell f(t) + f_k(t) + t^q$ has real coefficients all roots in $\mathbb{C} \setminus \mathbb{R}$ come in conjugate pairs. Since there is a unique unbounded root it follows that the root is real for large enough k . By the property that $f_k(t) + k^\ell f(t) + t^q$ has only non-negative coefficients it follows that all real roots are non-positive, hence the unbounded roots must go to $-\infty$ as $k \rightarrow \infty$. \square

Proof of Theorem 1.1. The assertion follows directly from Lemma 3.1 and Remark 2.5 if we set $q = \text{apd}(I)$, $f(t) = \frac{1}{(\ell(I)-1)!} \sum_{i=1}^{\mathfrak{R}(I)} k_i t^{\text{apd}(I)-i}$ and $f_k(t) = \mathfrak{P}(I)(k, t) - f(t) - t^{\text{apd}(I)}$. \square

A sequence a_0, \dots, a_q of real numbers is called log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for $1 \leq i \leq q-1$. We say that a non-necessarily log-concave sequence a_0, \dots, a_q is strictly log-concave at i if $a_i^2 > a_{i-1}a_{i+1}$. Log-concavity of a sequence of strictly positive numbers a_0, \dots, a_q implies that the sequence is unimodal, i.e. there is an i such that $a_0 \leq \dots \leq a_i \geq \dots \geq a_q$. This property is of interest in enumerative combinatorics and combinatorial commutative algebra. In the sequel we want to exhibit some facts that allow to deduce partial or full unimodality of the sequence $\beta_0(S/I^k), \dots, \beta_{\text{apd}(I)}(S/I^k)$ for large k .

The next remark identifies situations when we can expect strict log-concavity. The part (i) is a trivial consequence of the definition and part (ii) is a well know fact about real rooted polynomials (see for example [1] and the references therein).

Remark 3.2. (i) If a_0, \dots, a_q is a sequence of positive real numbers that is log-concave then there are numbers $0 \leq j_1 \leq j_2 \leq q$ such $a_0 < \dots < a_{j_1} = \dots = a_{j_2} > \dots > a_q$. In particular, a_0, \dots, a_q is strictly log-concave at i for $1 \leq i \leq j_1$ and $j_2 \leq i \leq q - 1$.

(ii) If $a_0 + a_1t + \dots + a_qt^q \in \mathbb{R}[t]$ has only real roots then a_0, \dots, a_q is log-concave.

Corollary 3.3. *Let I be a (graded) ideal in S . Assume that the coefficient series of $\sum_{i=1}^{\mathfrak{R}(I)} k_i \cdot t^{\text{apd}(I)-i}$ is strictly log-concave at $1 \leq i - 1, i, i + 1 \leq \text{apd}(I) - 2$. Then for large k the sequence $\beta_0(S/I^k), \beta_1(S/I^k), \dots, \beta_{\text{apd}(I)}(S/I^k)$ is strictly log-concave at i .*

Proof. Using the notation from Theorem 1.1 we set

$$b_k(t) = \frac{1}{(t - \gamma_k^{(\text{apd}(I))})} \mathfrak{P}(I)(k, t)$$

and $q = \text{apd}(I)$. Then $b_k(t)$ has roots converging to the roots of $\sum_{i=1}^{\mathfrak{R}(I)} k_i \cdot t^{\text{apd}(I)-i}$. Thus up to a constant factor the coefficients of $b_k(t)$ converge to the coefficients of $\sum_{i=1}^{\mathfrak{R}(I)} k_i t^{q-i}$. Since the coefficients are continuous in terms of roots this implies that the coefficient sequence of $b_k(t)$ is strictly log-concave for large k at $i - 1, i$ and $i + 1$. Now $\mathfrak{P}(I)(k, t) = \sum_{i=0}^q \mathfrak{P}_i(I)(k) t^{q-k}$ is obtained from $b_k(t)$ by multiplication with $(t - \gamma_k^{(q)})$. Set $\gamma := -\gamma_k^{(q)}$ and write $b_k(t) = c_0 + \dots + c_{q-2}t^{q-2} + c_{q-1}t^{q-1}$, where $c_{q-1} = 1$. If k is large enough and we set $c_{-1} = c_q = 0$ then $\beta_{q-i}(S/I^k) = \gamma c_i + c_{i-1}$ for $0 \leq i \leq q$. Hence strict log-concavity at $i - 1, i$ and $i + 1$ for large k implies::

$$\begin{aligned} \beta_{q-i}(S/I^k)^2 - \beta_{q-i-1}(S/I^k) \cdot \beta_{q-i+1}(S/I^k) &= (\gamma c_i + c_{i-1})^2 - \\ &\quad (\gamma c_{i-1} + c_{i-2})(\gamma c_{i+1} + c_i) \\ &= \gamma^2(c_i^2 - c_{i-1}c_{i+1}) + \\ &\quad \gamma(c_{i-1}c_i - c_{i-2}c_{i+1}) + c_{i-1}^2 - c_{i-2}c_i \\ &> \gamma(c_{i-1}c_i - c_{i-2}c_{i+1}) \end{aligned}$$

Multiplying $c_{i-1}c_i - c_{i-2}c_{i+1}$ by $c_{i-1}c_i$ we obtain $c_{i-1}^2c_i^2 - c_{i-2}c_{i-1}c_i c_{i+1}$. Again from strict log-concavity we know that $c_{i-1}^2 > c_{i-2}c_i$ and $c_i^2 > c_{i-1}c_{i+1}$. Since the coefficients of $b_k(t)$ are positive as they are up to a constant close to the coefficients of $\sum_{i=1}^{\mathfrak{R}(I)} k_i \cdot t^{q-i}$ it follows that $c_{i-1}^2c_i^2 - c_{i-2}c_{i-1}c_i c_{i+1} > 0$ and hence $c_{i-1}c_i - c_{i-2}c_{i+1} > 0$.

□

Example 3.4. Let I be generated by a regular sequence of length n . By using the Eagon-Northcott complex we see that $\beta_i(S/I^k) = \binom{k+n-1}{n-i} \binom{k-2+i}{i-1}$ for $1 \leq i \leq$

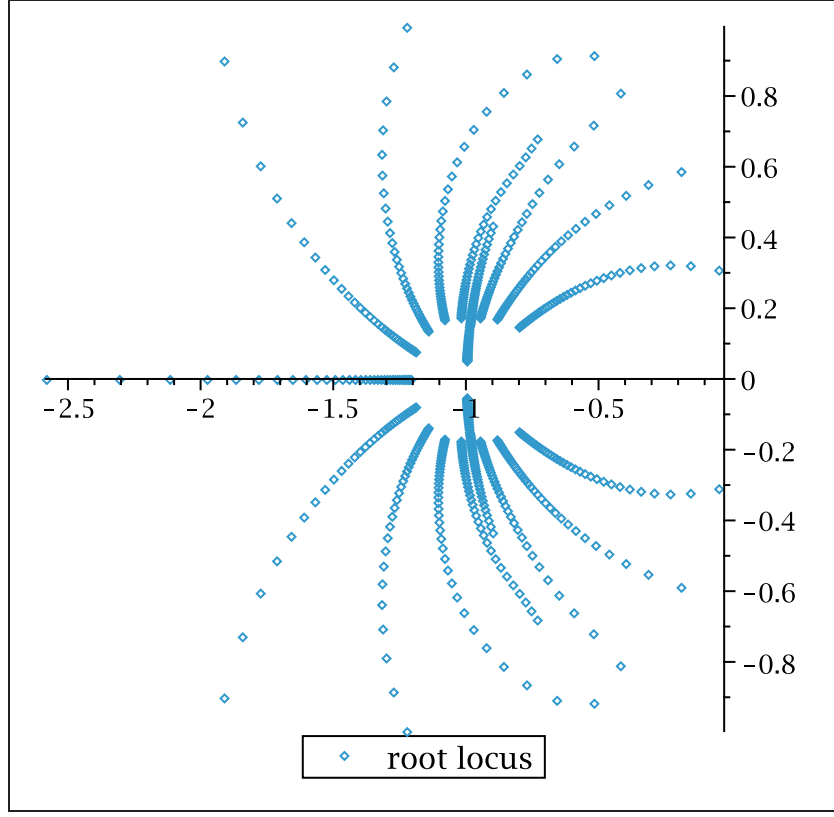


FIGURE 1. Root loci for Example 3.4 and parameters $n = 20$, $k \in \{1, \dots, 40\}$

$\text{apd}(I) = n$. Thus

$$\sum_{i=0}^q \mathfrak{P}_i(I)(k)t^{n-k} = t^n + \sum_{i=1}^n \binom{k+n-1}{n-i} \binom{k-2+i}{i-1} t^{n-i}.$$

In particular,

$$k_i = \frac{(n-1)!}{(n-i)!(i-1)!}$$

and therefore

$$\begin{aligned} \sum_{i=1}^n k_i t^{n-i} &= \sum_{i=1}^n \frac{(n-1)!}{(n-i)!(i-1)!} t^{n-i} \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} t^{n-1-i} \\ &= \frac{1}{(n-1)!} (1+t)^{n-1} \end{aligned}$$

Indeed this calculation is predicted by Corollary 2.8 when I is the maximal (graded) ideal in a polynomial ring. The calculation implies that all α_i from Theorem 1.1 are equal to -1 and the coefficient series is the sequence of binomial coefficients which is strictly log-concave. Hence Corollary 3.3 applies. Thus for large k the sequence $\beta_0(S/I^k), \dots, \beta_n(S/I^k)$ is strictly log-concave and hence unimodal. Clearly, this consequences of Corollary 3.3 can also be easily checked by inspection of the sequence $\beta_0(S/I^k) = 1, \beta_i(S/\mathfrak{m}^k) = \binom{k+n-1}{n-i} \binom{k-2+i}{i-1}, 1 \leq i \leq n$ in this case. This example also shows that the fact that all roots of $\sum_{i=1}^{\text{apd}(I)} k_i t^{\text{apd}(I)-i}$ are real does not force the roots of $\sum_{i=0}^n \mathfrak{P}_i(I)(k) t^{n-i}$ to be real for large k . Indeed, one can check that no root except for the two roots forced by Theorem 1.1 and depending on the parity of n one additional root of $\sum_{i=0}^q \mathfrak{P}_i(I)(k) t^{q-k}$ are real. In Figure 1 we have depicted the roots for $n = 20$ and k from 1 to 40 in this example with the imaginary axis being vertical and the real axis being horizontal. Indeed, the real root going to $-\infty$ is only seen for small k as it leaves the axis range already for small values of k . One easily recognizes the root curves converging to -1 in conjugate pairs.

Following the same argumentation as in Example 3.4 we deduce from Corollary 2.8 and Corollary 3.3 the last result of this paper.

Corollary 3.5. *Let $I \subset S$ be a monomial ideal generated in a single degree with $\dim S/I = 0$. Suppose that I has linear relations. Then for large k the sequence $\beta_0(S/I^k), \dots, \beta_n(S/I^k)$ is strictly log-concave and hence strictly unimodal.*

We do not know any ideal I for which the conclusion of Corollary 3.5 does not hold. But we do not see enough evidence to formulate a conjecture.

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