

Minimal Resolutions via Algebraic Discrete Morse Theory

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Abstract

Forman's discrete Morse theory is studied from an algebraic viewpoint. Analogous to independent work of Emil Sköldbberg, we show that this theory can be applied to chain complexes of free modules over a ring. We provide four applications of this theory:

- (i) We construct a new resolution of the residue field k over the k -algebra A , where $A = k[x_1, \dots, x_n]/\mathfrak{a}$ is the quotient of the commutative polynomial ring in n indeterminates by an ideal $\mathfrak{a}$. This resolution is a commutative analogue of the Anick resolution, which is a well studied resolution of k over quotient of the polynomial ring in non-commuting variables. We prove minimality of the resolution if $\mathfrak{a}$ admits a quadratic Gröbner basis or if $\text{in}_<(\mathfrak{a})$ is a complete intersection.
- (ii) Now let $A = k\langle x_1, \dots, x_n \rangle/\mathfrak{a}$ be the quotient of the polynomial ring in n non-commuting indeterminates by a two-sided ideal $\mathfrak{a}$. Sköldbberg shows how to construct the Anick resolution of A as well as the two-sided Anick resolution via algebraic discrete Morse theory. We derive the same result and prove, in addition, the minimality of these resolutions and the rationality of the Poincaré-Betti series in special cases.
- (iii) In the situation of (i) and (ii) we construct a resolution of A as an $(A \otimes A^{\text{op}})$ -module. For the same situation when the resolutions of k constructed in (i) and (ii) are minimal, we show that the resolution of A as an $(A \otimes A^{\text{op}})$ -module is minimal. Thereby we generalize a result of BACH used to calculate Hochschild homology in these cases.
- (iv) Let $S := k[x_1, \dots, x_n]$ be the commutative polynomial ring and $\mathfrak{a} \trianglelefteq S$ a monomial ideal. We construct a new minimal (cellular) free resolution of $\mathfrak{a}$ in the case, where $\mathfrak{a}$ is principal Borel fixed. Our resolution is a generalization of the hypersimplex resolution for powers of the maximal ideal, introduced by Batzies and Welker.

If $\mathfrak{a}$ is p -Borel fixed, a minimal free resolution is only known in the case where $\mathfrak{a}$ is principal Cohen-Macaulay. We construct minimal (cellular) free resolutions for a larger class of p -Borel fixed ideals. In addition, we give a formula for the multigraded Poincaré-Betti series and for the regularity. Our results generalize known results about regularity and Betti-numbers of p -Borel fixed ideals.

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CHAPTER 1

Introduction

Discrete Morse theory as developed by Forman [17],[18] allows to construct, starting from a (regular) CW-complex, a new homotopy equivalent CW-complex with fewer cells. In this paper we describe and apply an algebraic version of this theory, which we call *algebraic discrete Morse Theory*. Analogous theories were developed by Sköldbberg [40] and Kozlov [32]. We consider chain complexes $C_\bullet = (C_i, \partial_i)_{i \geq 0}$ of free modules C_i over a ring R . A priori, we always fix a basis of the C_i – the basis elements play the role of the cells in the topological situation. Then, by an application of algebraic discrete Morse theory one constructs a new chain complex of free R -modules such that the homology of the two complexes coincides. Before we come to the applications of algebraic discrete Morse theory which will contain the main results of this paper we first spend some time to outline the philosophy behind the theory.

Our construction looked upon from the point of algebra does the following. We are looking for collections $\{D_\bullet^{(i)} \mid i \in I\}$ of short exact subcomplexes

$$D_\bullet^{(i)} : 0 \rightarrow R \xrightarrow{\partial} R \rightarrow 0$$

in a given complex $C_\bullet$. We require:

- (A1) The direct sum $D_\bullet = \bigoplus_{i \in I} D_\bullet^{(i)}$ is a subcomplex of $C_\bullet$.

Then we calculate the quotient complex

$$\frac{C_\bullet}{D_\bullet}.$$

We require:

- (A2) The complex $\frac{C_\bullet}{D_\bullet}$ is a complex of free R -modules.

Thus, passing from $C_\bullet$ to $\frac{C_\bullet}{D_\bullet}$, we cancel out one copy of R in two consecutive homological degrees for each $D_\bullet^{(i)}$, $i \in I$. The description of the differential of the quotient complex is more subtle and will be provided later. Since each $D_\bullet^{(i)}$ is exact it follows that $D_\bullet$ is exact. Therefore, the homology of the quotient complex equals the homology of the complex $C_\bullet$.

Now we describe the combinatorial procedure that will reflect the algebraic construction described above. We proceed as follows: Consider the complex $C_\bullet$ as a directed weighted graph, where the vertex-set is given by the chosen basis of $C_\bullet$. Two vertices c, c' of homological degree i and $i - 1$ respectively are joined by a directed edge $e : c \rightarrow c'$ if c' appears with non-zero coefficient $[c : c'] \in R$ in the differential of c . The weight of the edge $e : c \rightarrow c'$ is then given by the coefficient $[c : c']$. Now we choose edges with invertible weights such that the collection of edges satisfies the following two conditions:

- (C1) The edges have pairwise disjoint vertex sets.

- (C2) Reversing the direction of the edges from the collection in the graph yields a graph that still contains no directed cycle.

Adapting the terminology from discrete Morse theory we call such a collection of edges an *acyclic matching* or a *Morse-matching* on $C_\bullet$.

Let us explain how the construction above reflects the algebraic approach described before. Assume that c and c' are vertices corresponding to basis elements in homological degrees i and $i - 1$ respectively such that c and c' are connected by an edge. Assume further that the weight on an edge between c and c' is invertible in R . Then we apply the basis transformation in the homological degree $i - 1$ that replaces c' by $\partial_i(c)$ and is the identity on all other basis elements. This yields an exact subcomplex

$$0 \rightarrow Rc \rightarrow R\partial_i(c) \rightarrow 0.$$

For a collection of subcomplexes of this type, their direct sum can be a subcomplex only if condition (C1) is satisfied. But (C1) is not sufficient. Now condition (C2) will be shown to imply the existence of a basis transformation that assures condition (A1). The fact that we have chosen edges that have invertible coefficients means that this basis transformation is indeed invertible over R and therefore (A2) is satisfied.

The advantage of this approach is that the calculation of the quotient-complex is purely combinatorial:

- ▷ The basis elements of the modules in the quotient-complex are in one-to-one correspondence with the vertices not lying in any edge $e \in \mathcal{M}$.
- ▷ The coefficient of the basis element c' in the differential $\partial^{\mathcal{M}}(c)$, taken in the quotient-complex, can be calculated by summing up the weights of all directed paths from c to c' in the graph where all edges from the matching are reversed. Here the weight of a path is the product of the weights of the edges along the path.

It should be mentioned that in practice the calculation of the differential of the quotient-complex can be very complicated and technical. But often it is enough to know the basis of the quotient-complex: For example, consider the problem of "finding Betti-numbers of an R -module": If we start with a multigraded free resolution then the multidegree of the remaining basis elements can give information about minimality of the resolution. Thus, in this situation for calculating Betti-numbers, regularity, and Poincaré-Betti series it is not necessary to calculate the differential.

Recall that Forman's discrete Morse theory (see [17], [18]) allows starting from a regular CW-complex the construction of a homotopy equivalent CW-complex with fewer cells. When applied to the complex calculating cellular homology of a regular CW-complex algebraic discrete Morse theory describes the impact of Forman's construction. As an advantage over Forman's theory algebraic discrete Morse theory can be applied iteratively: Discrete Morse theory can only be applied to a regular CW-complex, which implies that all coefficients of its cellular chain complex have to be ± 1 . Algebraic discrete Morse theory only requires certain coefficients to be invertible. Consider the following example. Assume we are given a regular CW-complex and want to calculate its homology with coefficients in a field of characteristic 0. If we apply Forman's theory to this complex we may arrive at a CW-complex for which certain face incidences give rise to coefficients $\neq \pm 1, 0$ in the cellular chain complex. Discrete Morse theory is no longer applicable to those

face incidences, whereas algebraic discrete Morse theory still can be applied to its cellular chain complex.

Often algebraic discrete Morse theory allows statements on minimal resolutions, Betti-numbers, regularity, etc., without explicitly constructing the minimal free resolution. For example, in [30] the first author constructs a sequence of acyclic matchings on the Taylor resolution of any monomial ideal $\mathfrak{a}$ in the polynomial ring S such that the resulting Morse complex is a minimal free resolution. Clearly, we cannot explicitly construct this resolution, but we retrieve information about its structure, which in special cases allows the calculation of the Betti-numbers of k viewed as an $S/\mathfrak{a}$ -module.

In this volume we present applications of algebraic discrete Morse theory to free resolutions of R -modules, where R is a quotient of the commutative or non-commutative polynomial ring in a finite number of variables.

We proceed as follows:

In Chapter 2 we describe how algebraic discrete Morse theory works and provide the basic constructions. We do not provide proofs in that chapter. Even though our version of algebraic discrete Morse theory is almost identical with the one developed by Sköldbberg [40] we do give proofs in Appendix B since our theory is formulated in a slightly more general framework and also contains a description of the differential of the resulting complex. The remaining chapters of the manuscript are devoted to applications of algebraic discrete Morse theory which comprise the central results of our work.

In Chapter 3 we consider resolutions of the field k over a quotient $A = S/\mathfrak{a}$ of the commutative polynomial ring $S = k[x_1, \dots, x_n]$ in n variables by an ideal $\mathfrak{a}$. We construct a free resolution of k as an A -module which can be seen as a generalization of the Anick resolution to the commutative case. We show that our resolution is minimal if $\mathfrak{a}$ admits a quadratic Gröbner basis. Similarly, we give an explicit description of the minimal free resolution of k , if there is a term order for which the initial ideal of $\mathfrak{a}$ is a complete intersection.

Chapter 4 considers the same situation in the non-commutative case. We apply algebraic discrete Morse theory in order to obtain the Anick resolution of the residue field k over $A = k\langle x_1, \dots, x_n \rangle / \mathfrak{a}$ from the normalized Bar resolution, where $k\langle x_1, \dots, x_n \rangle$ is the polynomial ring in n non-commuting indeterminates, and $\mathfrak{a}$ is a two-sided ideal. This result has also been obtained by Sköldbberg [40]. In addition we give a description of the differential. Using this description we get, in addition to [40], conditions on Gröbner bases, which imply minimality of the resolution. For example we prove the minimality of the resolution when $\mathfrak{a}$ is monomial or the Gröbner basis consists of homogeneous polynomials which all have the same degree. The first case provides an alternative proof of a result by Anick [1]. In the two cases, it follows that the Poincaré-Betti series is rational. This is well known when the ideal is monomial [4]. In particular, it follows that the Hilbert series of $A/\mathfrak{a}$ is rational if $\mathfrak{a}$ admits a quadratic Gröbner basis. Also, this fact is well known (see [39] or [3]). In the case when $A/\mathfrak{a} \cong k[\Lambda]$ is the (commutative) semigroup ring of an affine semigroup Λ and $\mathfrak{a}$ admits a quadratic Gröbner basis, then we derive a conjecture by Sturmfels [43] from our construction. Sturmfels had described for this type of ideals a complex of free A -modules which he conjectured to define a minimal free resolution of k as an A -module. Now it is easily seen that in this

case our resolution coincides with the one given by Sturmfels, which proves his conjecture. Indeed, the general construction was inspired by Sturmfels' conjecture.

In Chapter 5 we give a free resolution of A as an $(A \otimes A^{op})$ -module, where again $A = k\langle x_1, \dots, x_n \rangle / \mathfrak{a}$ and $(A \otimes A^{op})$ is the enveloping algebra of A over k . Using this resolution we obtain the minimal free resolution of $A = k[x_1, \dots, x_n] / \langle f_1, \dots, f_s \rangle$ as an $(A \otimes A^{op})$ -module, when there is a term order for which the initial ideal of $\langle f_1, \dots, f_s \rangle$ is a complete intersection. In case $\mathfrak{a} = \langle f \rangle$ is generated by a single polynomial such a construction was first given by BACH in [12]. As a consequence we are able to provide explicit description of the Hochschild homology of the rings in the cases, when our resolution is minimal, again generalizing results by BACH. There is recent work on the calculation of Hochschild homology when a Gröbner basis is known by Kobayashi [31]. Since this is not the first incident when Gröbner basis theory and discrete Morse theory meet, it appears to be worthwhile to work out the exact relation between the two approaches in general and in particular to the construction of minimal free resolution. We will not enlarge on this interesting topic in this volume and leave it as a challenging research problem.

Finally in Chapter 6 we construct minimal free resolutions for principal Borel fixed and for a class of p -Borel fixed ideals over the commutative polynomial ring $S := k[x_1, \dots, x_n]$. The resulting minimal free resolutions are even cellular.

In the first section of this chapter we recall the definition of a cellular resolution from [8] (see also [9]). Then we generalize the *hypersimplicial resolution* from [5] for powers of the homogeneous maximal ideal in the commutative polynomial ring, to a hypersimplicial resolution for principal Borel fixed ideals. Using Formans theory, Batzies and Welker [6] construct for the hypersimplicial resolution a discrete Morse matching, such that the resulting Morse complex is a minimal cellular free resolution of $\mathfrak{a}$. We show that the matching from [6] can be extended to the hypersimplicial resolution of principal Borel fixed ideals. Thus, we obtain a new minimal cellular resolution for principal Borel fixed ideals.

Note that minimal (cellular) resolutions for general Borel fixed ideals are known (see [6]) and an algebraic construction of a minimal free resolution was first given in [15]. For p -Borel fixed ideals minimal free resolutions were known only for principal Cohen-Macaulay ideals [2]. We construct minimal cellular resolutions for a larger class of p -Borel fixed ideals. In addition, we present a formula for the multigraded Poincaré-Betti series and for the regularity of those ideals. If the ideal is principal Cohen-Macaulay our resolution coincides with the known resolution. The results about the regularity generalize known facts.

In the Appendix A we derive the normalized Bar and Hochschild resolution as a simple application of algebraic discrete Morse theory and in Appendix B we give our proof of this theory. We have added Appendix A since it is a demonstration of algebraic discrete Morse theory in a relatively simple situation. Therefore, it can serve as a good example accompanying the presentation of algebraic discrete Morse theory in Chapter 2.

We think that there is wealth of further questions about minimal free resolution that can be answered using algebraic discrete Morse theory. Sköldberg for example calculates – using algebraic discrete Morse theory – the homology of the nilpotent Lie algebra, generated by $\{z, x_1, \dots, x_n, y_1, \dots, y_n\}$ with the only non-vanishing Lie bracket being $[z, x_i] = y_i$ over a field of characteristic 2 (see [40]). Therefore, we

hope that the results presented in this paper will provide motivation for studying and applying the theory to other contexts.

CHAPTER 2

Algebraic Discrete Morse Theory

In this chapter we describe an algebraic version of discrete Morse theory. Discrete Morse theory was developed in the mid 90's by Robin Forman (see [17], [18]). Starting from a regular CW-complex, discrete Morse theory provides a combinatorial tool for construction a homotopy equivalent CW-complex with fewer cells. Our theory is an algebraic version of Forman's theory which operates on chain complexes rather than on CW-complexes. It is generalization of results from [6] and an almost identical theory has been developed independently by Sköldbberg [40] and by Kozlov [32]. Our applications require a slightly more general setting than the one covered in [40] and [32]. Therefore, we give a detailed exposition of the theory here and provide proofs in Appendix B. We would like to note that meanwhile Emil Sköldbberg has developed an even more more general version of discrete Morse theory [42] that no longer requires free chain complexes.

Let R be a ring and $\mathbf{C}_\bullet = (C_i, \partial_i)_{i \geq 0}$ be a chain complex of free R -modules C_i . We choose a basis $X = \bigcup_{i \geq 0} X_i$ such that $C_i \simeq \bigoplus_{c \in X_i} R c$. From now on we write the differentials ∂_i with respect to the basis X in the following form:

$$\partial_i : \begin{cases} C_i & \rightarrow C_{i-1} \\ c & \mapsto \partial_i(c) = \sum_{c' \in X_{i-1}} [c : c'] \cdot c'. \end{cases}$$

Given the complex $\mathbf{C}_\bullet$ and the basis X , we construct a directed, weighted graph $G(\mathbf{C}_\bullet) = (V, E)$. The set of vertices V of $G(\mathbf{C}_\bullet)$ is the basis $V = X$ and the set E of (weighted) edges is given by the rule

$$(c, c', [c : c']) \in E \quad :\Leftrightarrow \quad c \in X_i, c' \in X_{i-1}, \text{ and } [c : c'] \neq 0.$$

We often omit the weight and write $c \rightarrow c'$ to denote an edge in E . Also by abuse of notation we write $e \in G(\mathbf{C}_\bullet)$ to indicate that e is an edge in E .

Definition 2.1. A finite subset $\mathcal{M} \subset E$ of the set of edges is called an acyclic matching if it satisfies the following three conditions:

- (1) (Matching) Each vertex $v \in V$ lies in at most one edge $e \in \mathcal{M}$.
- (2) (Invertibility) For all edges $(c, c', [c : c']) \in \mathcal{M}$ the weight $[c : c']$ lies in the center of R and is a unit in R .
- (3) (Acyclicity) The graph $G_{\mathcal{M}}(V, E_{\mathcal{M}})$ has no directed cycles, where $E_{\mathcal{M}}$ is given by

$$E_{\mathcal{M}} := (E \setminus \mathcal{M}) \cup \left\{ \left(c', c, \frac{-1}{[c : c']} \right) \text{ with } (c, c', [c : c']) \in \mathcal{M} \right\}.$$

For an acyclic matching $\mathcal{M}$ on the graph $G(\mathbf{C}_\bullet) = (V, E)$ we introduce the following notation, which is an adaptation of the notation introduced in [17] to our situation.

- (1) We call a vertex $c \in V$ critical with respect to $\mathcal{M}$ if c does not lie in any edge $e \in \mathcal{M}$; we write

$$X_i^{\mathcal{M}} := \{c \in X_i \mid c \text{ critical}\}$$

for the set of all critical vertices of homological degree i .

- (2) We write $c' \leq c$ if $c \in X_i$, $c' \in X_{i-1}$, and $[c : c'] \neq 0$.
 (3) $\text{Path}(c, c')$ is the set of paths from c to c' in the graph $G_{\mathcal{M}}(\mathbf{C}_{\bullet})$.
 (4) The weight $w(p)$ of a path $p = c_1 \rightarrow \dots \rightarrow c_r \in \text{Path}(c_1, c_r)$ is given by

$$w(c_1 \rightarrow \dots \rightarrow c_r) := \prod_{i=1}^{r-1} w(c_i \rightarrow c_{i+1}),$$

$$w(c \rightarrow c') := \begin{cases} -\frac{1}{[c : c']} & , \quad c \leq c', \\ [c : c'] & , \quad c' \leq c. \end{cases}$$

- (5) We write $\Gamma(c, c') = \sum_{p \in \text{Path}(c, c')} w(p)$ for the sum of weights of all paths from c to c' .

Now we are in position to define a new complex $\mathbf{C}_{\bullet}^{\mathcal{M}}$, which we call the Morse complex of $\mathbf{C}_{\bullet}$ with respect to $\mathcal{M}$. The complex $\mathbf{C}_{\bullet}^{\mathcal{M}} = (C_i^{\mathcal{M}}, \partial_i^{\mathcal{M}})_{i \geq 0}$ is defined by

$$C_i^{\mathcal{M}} := \bigoplus_{c \in X_i^{\mathcal{M}}} R c,$$

$$\partial_i^{\mathcal{M}} : \begin{cases} C_i^{\mathcal{M}} & \rightarrow \\ c & \mapsto \sum_{c' \in X_{i-1}^{\mathcal{M}}} \Gamma(c, c') c', \end{cases} .$$

Theorem 2.2. $\mathbf{C}_{\bullet}^{\mathcal{M}}$ is a complex of free R -modules and is homotopy equivalent to the complex $\mathbf{C}_{\bullet}$; in particular, for all $i \geq 0$

$$H_i(\mathbf{C}_{\bullet}) \cong H_i(\mathbf{C}_{\bullet}^{\mathcal{M}}).$$

The maps defined below are chain homotopies between $\mathbf{C}_{\bullet}$ and $\mathbf{C}_{\bullet}^{\mathcal{M}}$:

$$f : \begin{cases} \mathbf{C}_{\bullet} & \rightarrow \mathbf{C}_{\bullet}^{\mathcal{M}} \\ c \in X_i & \mapsto f(c) := \sum_{c' \in X_i^{\mathcal{M}}} \Gamma(c, c') c', \end{cases}$$

$$g : \begin{cases} \mathbf{C}_{\bullet}^{\mathcal{M}} & \rightarrow \mathbf{C}_{\bullet} \\ c \in X_i^{\mathcal{M}} & \mapsto g_i(c) := \sum_{c' \in X_i} \Gamma(c, c') c'. \end{cases}$$

The proof of Theorem 2.2 is given in Appendix B. Note that if $\mathbf{C}_{\bullet}$ is the cellular chain complex of a regular CW-complex and X is the set of cells of the regular CW-complex, then algebraic discrete Morse theory is the part of Forman's [17] discrete Morse theory which describes the impact of a discrete Morse matching on the cellular chain complex of the CW-complex.

Sometimes it is useful to consider the same construction for matchings which are not acyclic. Clearly, Theorem 2.2 does not hold anymore for $\mathbf{C}_{\bullet}^{\mathcal{M}}$ if $\mathcal{M}$ is not acyclic. In general, there is not even a good definition of the differentials $\partial_i^{\mathcal{M}}$.

But for calculating invariants it is nonetheless sometimes useful to consider $C_\bullet^{\mathcal{M}}$ for matchings that are not acyclic. Thus, if $\mathcal{M}$ is a not necessarily acyclic matching we denote by $C_\bullet^{\mathcal{M}}$ the corresponding sequences of free R -modules. For applications of this viewpoint see also [30].

Finally, we would like to generalize the construction of the Morse complex to infinite acyclic matchings:

Note, that the definition of an acyclic matching makes perfect sense also for infinite sets of edges. But if $\mathcal{M}$ is an infinite acyclic matching then $\Gamma(c, c')$ may no longer be well defined in case the set of paths from c to c' is infinite. Moreover, we indeed use finiteness in our proof of Theorem 2.2 since we use induction on the cardinality of the acyclic matching.

In order to be able to formulate a result similar to Theorem 2.2 for infinite acyclic matchings we have to introduce an additional finiteness condition.

Let $C_\bullet$ be a complex and $\mathcal{M}$ an infinite acyclic matching. Clearly the matching $\mathcal{M}$ induces a finite matching on each finite subcomplex $C_\bullet^f$ of $C_\bullet$. Therefore, we make the following definition:

Definition 2.3 (Finiteness). Let $C_\bullet$ be a complex of free R -modules and let $\mathcal{M}$ be an infinite acyclic matching. We say that $\mathcal{M}$ defines a Morse matching if there exists a sequence of finite subcomplexes $D_i := (D_\bullet)_i$, $i \geq 0$ of $C_\bullet$ such that:

- (1) D_i is a subcomplex of D_{i+1} , for all $i \geq 0$.
- (2) $C_\bullet = \text{colim}_{i \geq 0} D_i$.
- (3) $(D_i)^{\mathcal{M}}$ is a subcomplex of $(D_{i+1})^{\mathcal{M}}$.

Note that the last condition implies that $\Gamma(c, c')$ is well defined and thus the conclusion of Theorem 2.2 still holds for those infinite Morse matchings.

In our applications all complexes are multigraded by $\alpha \in \mathbb{N}^n$ and the α -graded part of $C_\bullet$ is finite. Therefore, the subcomplexes D_i , defined by

$$D_i := \bigoplus_{j=0}^i \bigoplus_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha|=j}} (C_\bullet)_\alpha$$

are finite subcomplexes. It is easy to see that for multigraded complexes whose graded parts are of finite rank any acyclic matching fulfills the additional finiteness condition on the sequence D_i . This indeed holds for all complexes in our applications.

Therefore, we get:

Theorem 2.4. *Let $C_\bullet$ be a $\mathbb{N}^n$ -graded complex of free R -modules such that $(C_\bullet)_\alpha$ is a finite subcomplex for all $\alpha \in \mathbb{N}^n$. Then the conclusion of Theorem 2.2 still holds for infinite acyclic matchings $\mathcal{M}$.*

Before we proceed to the applications, we give a simple example that shows that the finiteness assumption of Theorem 2.4 indeed is needed.

Example 2.5. Let $C_\bullet$ be the (non-reduced) chain complex of the simplicial complex triangulating the real line by 1-simplices $[i, i+1]$ for $i \in \mathbb{Z}$. Then C_0 and C_1

are free $\mathbb{Z}$ -modules of countably infinite dimension. It is easily seen that matching each 0-cell $\{i\}$ with the 1-cell $[i, i+1]$ yields an acyclic matching. The corresponding Morse complex is constant 0. In particular, its homology in dimension 0 is 0 being different from the 0-th (non-reduced) homology group of the real line, which is $\mathbb{Z}$.

In the following chapters we will use the conclusions of Theorem 2.2 and 2.4 in order to construct minimal free resolutions without explicitly referring to the theorems.

CHAPTER 3

Resolution of the Residue Field in the Commutative Case

Let $A = S/\mathfrak{a}$ be the quotient algebra of the commutative polynomial ring $S = k[x_1, \dots, x_n]$ in n indeterminates by the ideal $\mathfrak{a} \trianglelefteq S$.

The aim of this chapter is to deduce via algebraic discrete Morse theory a new free resolution of the residue field $k \cong A/\langle x_1, \dots, x_n \rangle$ as an A -module from the normalized Bar resolution. We write $\mathbf{NB}_\bullet^A = (B_i, \partial_i)_{i \geq 0}$ for the normalized Bar resolution of k over A (see Appendix A.1 or [47]).

In Section 1 we introduce basic facts about (commutative) Gröbner basis and relate Gröbner basis and initial ideals to discrete Morse theory. In Section 2 we show how in this context discrete Morse theory can be applied to derive an analog of the Anick-resolution for commutative rings. Then in Section 3 we study conditions that imply that this commutative Anick-resolution is minimal. These cases include ideals with quadratic Gröbner basis and with initial ideals that are complete intersections.

1. Gröbner Bases and Discrete Morse Theory

From now on let $\mathfrak{a} = \langle f_1, \dots, f_s \rangle \trianglelefteq S$ be an ideal, such that the set $\{f_1, \dots, f_s\}$ is a minimal reduced Gröbner basis with respect to a fixed degree-monomial order $\prec$ (for example degree-lex or degree-revlex). We assume that $x_1 \succ x_2 \succ \dots \succ x_n$ and we write $\mathcal{SM}$ for the corresponding set of standard monomials of degree ≥ 1 .

It is well known that $\mathcal{SM} \cup \{1\}$ is a basis of A as k -vector space. Thus, for any monomial w in S there is a unique representation

$$(3.1) \quad w = a_1 + \sum_{v \in \mathcal{SM}} a_v v, \quad a_1, a_v \in k,$$

as a linear combination of standard monomials in A .

Since we assume that our monomial order is a refinement of the degree order on monomials, it follows that $a_v = 0$ for $|v| > |w|$. Here we denote with $|v|$ the total degree of the monomial v . In this situation we say that v is reducible to $-\sum_{v \in \mathcal{SM}} a_v v$. Note that since we use the normalized Bar resolution, the summand a_1 can be omitted.

Using the described reduction process we write the normalized Bar resolution $\mathbf{NB}_\bullet^A = (B_i, \partial_i)$ as

$$\begin{aligned} B_0 &:= A, \\ B_i &:= \bigoplus_{w_1, \dots, w_i \in \mathcal{SM}} A[w_1 | \dots | w_i], \quad i \geq 1 \end{aligned}$$

with differential

$$\begin{aligned} \partial_i([w_1 | \dots | w_i]) &= w_1 [w_2 | \dots | w_i] \\ &+ \sum_{j=1}^{i-1} (-1)^j \sum_{v \in \mathcal{SM}} a_{jv} [w_1 | \dots | w_{j-1} | v | w_{j+2} | \dots | w_i], \end{aligned}$$

for $w_j w_{j+1} = a_{j,1} + \sum_{v \in \mathcal{SM}} a_{jv} v$, with $a_{jv} \in k$ and $v \in \mathcal{SM}$.

The following convention will be convenient. For a monomial $w \in S$ we set $m(w) := \min\{i \mid x_i \text{ divides } w\}$. Finally, we think of $[w_1 | \dots | w_i]$ as a vector, and we speak of w_j as the entry in the j th coordinate position.

Now we describe the acyclic matching on the normalized Bar resolution, which will be crucial for the proof of Theorem 3.6. Since all coefficients in the normalized Bar resolutions are ± 1 , condition (2) of Definition 2.1 is automatically fulfilled. Thus, we only have to take care of the conditions (1) and (3).

We inductively define acyclic matchings $\mathcal{M}_j$, $j \geq 1$, that are constructed with respect to the j th coordinate position. We start with the leftmost coordinate position $j = 1$. We set

$$\mathcal{M}_1 := \left\{ \begin{array}{c} [x_{m(w_1)} | w'_1 | w_2 | \dots | w_l] \\ \downarrow \\ [w_1 | w_2 | \dots | w_l] \end{array} \in G(\mathbf{NB}_\bullet^A) \mid w_1 = x_{m(w_1)} w'_1 \right\}.$$

The set of critical cells $B_l^{\mathcal{M}_1}$ in homological degree $l \geq 1$ is given by:

- (1) $B_1^{\mathcal{M}_1} := \{[x_i] \mid 1 \leq i \leq n\}$,
- (2) For $l \geq 2$, $B_l^{\mathcal{M}_1}$ is the set of all $[x_i | w_2 | w_3 | \dots | w_l]$, $w_2, \dots, w_l \in \mathcal{SM}$ that satisfy either
 - $\rightarrow i \leq m(w_2)$ and $x_i w_2$ is reducible or
 - $\rightarrow i > m(w_2)$.

Assume now $j \geq 2$ and $\mathcal{M}_{j-1}$ is defined. Let $\mathcal{B}^{\mathcal{M}_{j-1}}$ be the set of critical cells left after applying $\mathcal{M}_1 \cup \dots \cup \mathcal{M}_{j-1}$.

Let $\mathcal{E}_j$ denote the set of edges in $G(\mathbf{NB}_\bullet^A)$ that connect critical cells in $\mathcal{B}^{\mathcal{M}_{j-1}}$. The following condition on an edge in $\mathcal{E}_j$ defines the matching $\mathcal{M}_j$.

Definition 3.1 (Matching Condition). Let

$$\begin{array}{c} [x_{i_1} | w_2 | \dots | w_{j-1} | u_1 | u_2 | w_{j+1} | \dots | w_l] \\ \downarrow \\ [x_{i_1} | w_2 | \dots | w_{j-1} | w_j | w_{j+1} | \dots | w_l] \end{array}$$

be an edge in $\mathcal{E}_j$. In particular, $w_j = u_1 u_2$. We say that the edge satisfies the matching condition if u_1 is the maximal monomial with respect to $\prec$ such that

- (i) u_1 divides w_j ,
- (ii) $[x_{i_1} | w_2 | \dots | w_{j-1} | u_1 | u_2 | w_{j+1} | \dots | w_l] \in \mathcal{B}^{\mathcal{M}_{j-1}}$,
- (iii) $[x_{i_1} | w_2 | \dots | w_{j-1} | v_1 | v_2 | w_{j+1} | \dots | w_l] \notin \mathcal{B}^{\mathcal{M}_{j-1}}$ for each $v_1 \mid u_1$, $v_1 \neq u_1$ and $v_1 v_2 = w_j$.

$$\mathcal{M}_j := \left\{ \begin{array}{c} [x_{i_1} | w_2 | \dots | w_{j-1} | u_1 | u_2 | w_{j+1} | \dots | w_l] \\ \downarrow \\ [x_{i_1} | w_2 | \dots | w_{j-1} | w_j | w_{j+1} | \dots | w_l] \end{array} \in \mathcal{E}_j \text{ satisfying 3.1} \right\}.$$

We write $\text{MinGen}(\text{in}_{\prec}(\mathfrak{a}))$ for the minimal, monomial generating system of the initial ideal of $\mathfrak{a}$ with respect to the chosen monomial order $\prec$. The set of critical cells $B_l^{\mathcal{M}_j}$ in homological degree $l \geq 1$ is given by

- (1) $B_1^{\mathcal{M}_j} := \{[x_i] \mid 1 \leq i \leq n\}$,
- (2) $B_2^{\mathcal{M}_j}$ consists of elements $[x_i|w_2]$ such that either $w_2 = x_{i'}$ for some i' and $i > i'$ or $x_i w_2 \in \text{MinGen}(\text{in}_{\prec}(\mathfrak{a}))$,
- (3) For $l \geq 3$, $B_l^{\mathcal{M}_j}$ consists of elements $[x_i|w_2|\dots|w_j|\dots|w_l] \in \mathcal{B}_l^{\mathcal{M}_{j-1}}$, such that for each divisor $u \mid w_j$ we have $[x_i|w_2|\dots|w_{j-1}|u|\dots|w_l] \notin \mathcal{B}_l^{\mathcal{M}_{j-1}}$ and one of the following conditions is satisfied:
 - $\rightarrow w_j w_{j+1}$ is reducible or
 - $\rightarrow w_j w_{j+1} = uv \in \mathcal{SM}$ and
 - $[x_i|w_2|\dots|w_{j-1}|u|v|w_{j+2}|\dots|w_l] \in \mathcal{B}_l^{\mathcal{M}_{j-1}}$,
 - $u \succ w_j$,
 - $[x_i|w_2|\dots|w_{j-1}|u'|v'|w_{j+2}|\dots|w_l] \notin \mathcal{B}_l^{\mathcal{M}_{j-1}}$ for each divisor $u' \mid u$, $u' \neq u$ and $u'v' = w_j w_{j+1}$.

We finally set $\mathcal{M} := \bigcup_{j \geq 1} \mathcal{M}_j$ and we write $\mathcal{B}^{\mathcal{M}}$ for the set of critical cells with respect to $\mathcal{M}$.

Lemma 3.2. *$\mathcal{M}$ is an acyclic matching.*

PROOF. Since all coefficients are ± 1 , the condition (2) of Definition 2.1 is automatic. Property (1) is satisfied by definition of $\mathcal{M}$.

It remains to verify the acyclicity condition (3). Assume that there exists a directed cycle. Let

$$w = [m_1|\dots|m_l|m_{l+1}|\dots|m_i] \rightarrow v = [m_1|\dots|m_l m_{l+1}|\dots|m_i]$$

be the downward edge in that cycle for which l is minimal and for which m_l is of minimal degree among all downward edges with parameter l . By minimality of l we know that any upward edge

$$[n_1|\dots|n_{j-1}|n_j|n_{j+1}|\dots|n_{i-1}] \rightarrow [n_1|\dots|n_{j-1}|v_1|v_2|n_{j+1}|\dots|n_{i-1}]$$

in the cycle satisfies $j \geq k$ and $n_1 = m_1, \dots, n_{k-1} = m_{k-1}$. Since the path starting in w returns to w , there must be a first downward edge

$$v' = [m_1|\dots|m_l m_{l+1} u|m'_{l+1}|\dots|m'_i] \rightarrow w' = [m_1|\dots|v_1|v_2|m'_{l+1}|\dots|m'_i]$$

that is split in the l th coordinate. By minimality of m_l we know that m_l divides v_1 . By induction on the downward edges of that type this shows that there is an edge

$$[m_1|\dots|m'_l|m''_{l+1}|\dots|m''_i] \rightarrow [m_1|\dots|m_l|u_2|m''_{l+1}|\dots|m''_i]$$

in the cycle. By property (iii) of Definition 3.1 it follows that $v_1 = m_l$. Since the matching is built inductively by increasing coordinates it follows that the edge

$$[m_1|\dots|m_l m_{l+1} u|m'_{l+1}|\dots|m'_i] \rightarrow [m_1|\dots|v_1|v_2|m'_{l+1}|\dots|m'_i]$$

must immediately follow the edge

$$[m_1|\dots|m_l|m_{l+1}|\dots|m_i] \rightarrow [m_1|\dots|m_l m_{l+1}|\dots|m_i]$$

But then $w = w'$ and $v = v'$ which yields a contradiction. $\square$

2. An Anick Resolution for the Commutative Polynomial Ring

In this section we look closer into the Morse complex corresponding to the acyclic matching $\mathcal{M}$ from Lemma 3.2. For this we choose the degree-lex order as our fixed monomial order. We write $\text{MinGen}(\text{in}_{\prec}(\mathfrak{a}))$ for the minimal, monomial generating system of the initial ideal of $\mathfrak{a}$ with respect to degree-lex.

In order to describe the critical cells for the chosen term order, we first define the concept of a minimal fully attached tuple. Note that the notation *fully attached* was introduced by Sturmfels (see Example 4.10 and [43]). Our definition was inspired by Sturmfels' construction.

Definition 3.3. A pair $[w_1|w_2]$ is called minimal fully attached if

$$w_1 = x_{m(w_1w_2)} \text{ and } w_1w_2 \in \text{MinGen}(\text{in}_{\prec}(\mathfrak{a})).$$

Assume $l > 2$. An l -tuple $[w_1|\dots|w_{l-1}|w_l]$ is called minimal fully attached if $[w_1|\dots|w_{l-1}]$ is minimal fully attached, $m(w_1) \leq m(w_j)$ for $j = 3, \dots, l$, and one of the following conditions is satisfied:

- (1) $w_{l-1}w_l$ is reducible or
- (2) $w_{l-1}w_l = uv \in \mathcal{SM}$ with $u \succ w_{l-1}$ and $[w_1|\dots|w_{l-2}|u]$ is a minimal fully attached $(l-1)$ -tuple,

and w_l is the minimal monomial such that no divisor $w'_l \mid w_l$, $w'_l \neq w_l$, satisfies one of the two conditions above.

In the following we check that the basis of the free modules in the Morse complex $\text{NB}_{\bullet}^{\mathcal{M}}$ is given as the set $\mathcal{B}$ of words over the alphabet $\Sigma = \Sigma_1 \cup \Sigma_2$, where

$$\begin{aligned} \Sigma_1 &:= \left\{ [x_{i_1}|x_{i_2}|\dots|x_{i_r}] \mid 1 \leq i_r < i_{r-1} < \dots < i_1 \leq n \right\}, \\ \Sigma_2 &:= \left\{ [x_{w_2}|w_2|\dots|w_l] \mid [x_{w_2}|w_2|\dots|w_l] \text{ minimal fully attached} \right\} \end{aligned}$$

that contain none of the words:

$$(3.2) \quad [x_{i_1}|\dots|x_{i_r}][x_{w_2}|w_2|\dots|w_l], \quad x_{w_2} \preceq x_{i_r},$$

$$(3.3) \quad [x_{i_1}|\dots|x_{i_r}][x_{j_1}|\dots|x_{j_s}], \quad x_{j_1} \preceq x_{i_r},$$

$$(3.4) \quad [x_{w_2}|w_2|\dots|w_l][x_{i_1}|\dots|x_{i_r}], \quad x_{i_1} \prec x_{w_2},$$

$$(3.5) \quad [x_{w_2}|w_2|\dots|w_l][x_{v_2}|v_2|\dots|v_l], \quad x_{v_2} \prec x_{w_2}.$$

In order to be able to identify elements of $\mathcal{B}$ as basis elements of the Bar resolution, we read in a word from $\mathcal{B}$ the sequence of letters '[' as '['. If this convention is applied then any element of $\mathcal{B}$ can be read as some $[w_1|\dots|w_j]$ and corresponds to a basis element in some homological degree j . We collect the elements from $\mathcal{B}$ which are of homological degree j in $\mathcal{B}_j$ and call an element of $\mathcal{B}$ a fully attached tuple. We claim that there is a bijection between $\mathcal{B}^{\mathcal{M}}$ and $\mathcal{B}$ preserving the homological degree. To see this, consider a fully attached tuple $[x_{i_1}|w_2|\dots|w_i]$. Then the definition of a fully attached tuple implies that either $w_2 = x_s$ with $x_s \succ x_{i_1}$ (resp. $i_1 > s$) or $x_{i_1}w_2 \in \text{MinGen}(\text{in}_{\prec}(\mathfrak{a}))$. In the first case we split the tuple into $[x_{i_1}][x_s|w_3|\dots|w_i]$. We inductively apply this process to $[x_s|w_3|\dots|w_i]$ and merge the split off variables. We finally obtain

$$[x_{i_1}|x_{i_2}|\dots|x_{i_r}][x_{v_2}|v_2|\dots|v_s]$$

with $i_1 > \dots > i_r$, $x_{i_r} \prec x_{v_2}$, and $x_{v_2}v_2 \in \text{MinGen}(\text{in}_{\prec}(\mathfrak{a}))$. Indeed, the rules (3.2) and (3.3) are made to cope with this process. Now consider $[x_{v_2}|v_2|\dots|v_s]$. Then the definition of a fully attached tuple implies that either $v_3 = x_j$ with $x_j \succeq x_{v_2}$ or $x_{m(v_3)} \prec x_{v_2}$. In the first case we split the tuple into

$$[x_{v_2}|v_2][x_j|v_4|\dots|v_i],$$

. In the second case the monomial v_4 satisfies the same conditions as v_3 , so we split into

$$[x_{v_2}|v_2|v_3][x_j|v_5|\dots|v_i].$$

By construction $[x_{v_2}|v_2|v_3]$ is a minimal fully attached tuple and the conditions on v_3 and x_j explain the rules (3.4) and (3.5). If we continue this process, we obtain exactly the words in $\mathcal{B}$.

Remark 3.4. Let $\mathcal{L}$ be the language of all word over the alphabet Σ_2 that contain none of the words (3.5) as factors. To a letter $[x_{i_1}|x_{i_2}|\dots|x_{i_r}] \in \Sigma_1$ with $1 \leq i_r < i_{r-1} < \dots < i_1 \leq n$, we associate the symbol $e_{\{i_r < i_{r-1} < \dots < i_1\}}$. For $w \in \mathcal{B}^{\mathcal{M}}$ such that $w = e_{I_1} \dots e_{I_s}$, rule (3.3) shows that this word is considered as a basis element of $\text{NB}_{\bullet}^{\mathcal{M}}$, equivalent to the symbol $e_{I_1 \cup \dots \cup I_s}$. Therefore, an arbitrary word $w \in \mathcal{B}^{\mathcal{M}}$ can be rewritten as

$$w_1 e_{I_1} w_2 e_{I_2} \dots w_s e_{I_s}.$$

The rules (3.2) and (3.4) imply that the sets I_i are pairwise disjoint and in a decreasing order. Therefore, as a basis element of $\text{NB}_{\bullet}^{\mathcal{M}}$ the word w is equivalent to

$$e_{I_1 \cup \dots \cup I_s} w_1 w_2 \dots w_s.$$

It follows that we have a degree-preserving bijection between $\mathcal{B}^{\mathcal{M}}$ and the set

$$\left\{ e_I \mathbf{w} \mid I \subset \{1, \dots, n\} \text{ and } \mathbf{w} \in \mathcal{L} \right\}.$$

We will use this fact later in order to calculate the multigraded Poincaré-Betti series of k over A (see Corollary 3.8).

In order to describe the differential of the Morse complex, we introduce three reduction rules for fully attached tuples. These reduction rules will be based on the unique Gröbner representation (3.1) which will play the role of the basic set of rules:

$$\mathcal{R} := \left\{ v_1 v_2 \xrightarrow{a_w} w \mid \begin{array}{l} v_1, v_2 \in \mathcal{SM} \\ v_1 v_2 \notin \mathcal{SM} \end{array} \text{ and } \begin{array}{l} v_1 \cdot v_2 = a_0 + \sum_{w \in \mathcal{SM}} a_w w, \\ a_w \in k \end{array} \right\}.$$

Note that $w \xrightarrow{0} 0 \in \mathcal{R}$ is allowed (it happens if one of the generators f_i is a monomial).

Definition 3.5. Let $e_1 := [w_1|\dots|w_{i-1}|w_i|w_{i+1}|w_{i+2}|\dots|w_l]$ be an l -tuple of standard monomials.

Type I: Assume $[w_1|\dots|w_i]$ is fully attached. We say e_1 can be reduced to $e_2 := [w_1|\dots|w_{i-1}|v_i|w_{i+1}|w_{i+2}|\dots|w_l]$ if

- (i) $[w_1|\dots|w_{i-1}|v_i]$ is fully attached,
- (ii) $v_i v_{i+1} \in \mathcal{SM}$,
- (iii) $w_i w_{i+1} \xrightarrow{a} v_i v_{i+1} \in \mathcal{R}$ with $a \neq 0$.

In this case we write $e_1 \xrightarrow{-a}_1 e_2$.

Type II: We say that e_1 can be reduced to $e_2 := [w_1|\dots|w_{i-1}|v|w_{i+2}|\dots|w_l]$ if

- (i) $w_i w_{i+1} \xrightarrow{a} v \in \mathcal{R}$ with $a \neq 0$ and
- (ii) e_2 is a fully attached $(l-1)$ -tuple.

In this case we write $e_1 \xrightarrow{(-1)^i a} {}_2 e_2$.

Type III: We say that e_1 can be reduced to e_2 with coefficient $c := w_1$ (we write $e_1 \xrightarrow{w_1} e_2$) if $|w_2| \geq 2$ and $e_2 := [x_{m(w_2)} | w_2 / x_{m(w_2)} | w_3 | \dots | w_l]$.

Now let $e = [w_1 | \dots | w_l]$ and $f = [v_1 | \dots | v_{l-1}]$ be fully attached l - and $(l-1)$ -tuples. We say that e can be reduced to f with coefficient c ($e \xrightarrow{c} f$) if there exists a sequence $e = e_0, e_1, \dots, e_{r-1}$ and either

- (1) an e_r with $e_r = [u | v_1 | \dots | v_{l-1}] = [u | f]$ such that e_0 can be reduced to e_r with reductions of Type I and III, i.e.

$$e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r;$$

in this case we set $c := ((-1)^r \prod_{i=1}^r a_i) u$, or

- (2) an e_r such that e_0 can be reduced to e_r with reductions of Type I and III and e_r can be reduced to f with the reduction of Type II, i.e.

$$e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r \xrightarrow{(-1)^j b} f;$$

in this case we set $c := (-1)^{r+j} \cdot b \cdot \prod_{i=1}^r a_i$.

There may be several possible reduction sequences leading from e to f and the reduction coefficient may depend on the chosen sequence. We define *the* reduction coefficient $[e : f]$ to be the sum over all possible sequences. If there exists no sequence, we set $[e : f] := 0$.

The complex $F_\bullet$ is then given by

$$F_j := \bigoplus_{e \in \mathcal{B}_j} A e,$$

$$\begin{aligned} \partial : F_i &\rightarrow F_{i-1} \\ e &\mapsto \sum_{f \in \mathcal{B}_{i-1}} [e : f] f. \end{aligned}$$

Now we have:

Theorem 3.6. $F_\bullet = (F_\bullet, \partial)$ is an A -free resolution of the residue field k , which is minimal if and only if no reduction of Type II is possible.

PROOF. The fully attached tuples are exactly the critical cells. The reduction rules describe the Morse differential: As seen before, we have

$$\partial^{\mathcal{M}}([w_1 | \dots | w_l]) := w_1 [w_2 | \dots | w_l] + \sum_{i=1}^{l-1} (-1)^i [w_1 | \dots | w_i w_{i+1} | \dots | w_l]$$

If $[w_2 | \dots | w_l] \notin \mathcal{B}$, we have $[w_2 | \dots | w_l] = \partial([x_{i_2} | w'_2 | w_3 | \dots | w_l])$, which is described by the reduction of Type III.

For $[w_1 | \dots | w_i w_{i+1} | \dots | w_l]$ we have to distinguish three cases:

- (Case 1) $[w_1 | \dots | v_{ij} | \dots | w_l]$ is critical. Then we have $w_{i-1} v_{ij}$, $v_{ij} w_{i+2}$ reducible and $w_{i-1} u_1 v_{ij} u_2 \in \mathcal{SM}$ for all divisors u_1 of v_{ij} and u_2 of w_{i+2} . This situation is described by the reduction of Type II.

(Case 2) $[w_1 | \dots | v_{ij} | \dots | w_l]$ is matched by a higher degree cell. Then we have $w_{i-1}u_1$ reducible for $v_{ij} = u_1u_2$, and for all divisors u' of u_1 the monomial $w_{i-1}u'$ lies in $\mathcal{SM}$. Then we have

$$[w_1 | \dots | v_{ij} | \dots | w_l] = (-1)^{i+1} [w_1 | \dots | w_{i-1} | u_1 | u_2 | w_{i+2} | \dots | w_l],$$

which is a reduction of Type I.

(Case 3) $[w_1 | \dots | v_{ij} | \dots | w_l]$ is matched by a lower degree cell. In this case we have $[w_1 | \dots | v_{ij} | \dots | w_l] = 0$.

The coefficients of the reductions are exactly the coefficients of the Morse differential. Hence the Morse differential induces a sequence of reductions of Type I and III with either a reduction of Type II or the map $e_r = [v_1 | \dots | v_l] \xrightarrow{v_1} [v_2 | \dots | v_l]$ at the end, which gives our definition of the reduction coefficient. $\square$

Remark 3.7. In Chapter 4, we will see that in the non-commutative case our matching on the normalized Bar resolution gives the Anick resolution (for the definition, see [1]). Therefore, one can understand the resolution $F_\bullet$ as a generalization of the Anick resolution to the commutative polynomial ring.

If A is endowed with the natural multigrading $\deg(x_i) = e_i \in \mathbb{N}^n$, the multigraded Poincaré-Betti series of k over A is defined to be

$$P_k^A(\underline{x}, t) := \sum_{\substack{i \geq 0 \\ \alpha \in \mathbb{N}^n}} \dim_k(\text{Tor}_i^A(k, k)_\alpha) \underline{x}^\alpha t^i.$$

Remark 3.4 implies:

Corollary 3.8. *The Poincaré-Betti series of A satisfies*

$$P_k^A(\underline{x}, t) \leq \prod_{i=1}^n (1 + x_i t) F(\underline{x}, t),$$

where $F(\underline{x}, t) := \sum_{w \in \mathcal{L}} w t^{|w|}$ counts the words $w \in \mathcal{L}$. Here w is treated as the monomial in $x_1, \dots, x_n$ and $|w|$ denotes the length of w .

The inequality is an inequality between the coefficients of the power series expansion. $\square$

3. Two Special Cases

First, we consider a subclass of the class of Koszul algebras. We refer the reader to [39] and [19] for background on Koszul algebras. It is well known that if $\mathfrak{a}$ has a quadratic Gröbner basis then $A = S/\mathfrak{a}$ is Koszul. It is easy to see that in this case the minimal fully attached tuples have the following form: $[x_{i_1} | x_{i_2} | \dots | x_{i_r}]$. Therefore, a reduction of Type II is not possible and we get:

Corollary 3.9. *If $A = S/\mathfrak{a}$ and $\mathfrak{a}$ admits a quadratic Gröbner basis, then the resolution $F_\bullet$ is minimal.* $\square$

To get an explicit form of the multigraded Poincaré-Betti series in this case, one only has to calculate the word-counting function $F(\underline{x}, t)$ of the language $\mathcal{L}$. In this case, the multigraded Poincaré-Betti series coincides with the multigraded Poincaré-Betti series of $S/\text{in}_<(\mathfrak{a})$. Since the Poincaré-Betti series of monomial rings are studied by the first author in a more general context (see [30], [11]), where explicit formulas for the Poincaré-Betti series are obtained, we do not give the explicit form here.

As a second example where our resolution is minimal we would like to discuss the following situation:

Let $\mathfrak{a} = \langle f_1, \dots, f_s \rangle \trianglelefteq S$ be an ideal, such that $f_1, \dots, f_s$ is a reduced Gröbner basis with respect to the degree-lex order and such that the initial ideal $\text{in}_<(\mathfrak{a})$ is a complete intersection. Assume $f_j = m_j + \sum_{\alpha \in \mathbb{N}^n} f_{j\alpha} x^\alpha$ with leading monomial m_j . Since $\text{in}_<(\mathfrak{a})$ is a complete intersection, there exist exactly s minimal fully attached tuples, namely $t_i := \left[x_m(m_i) \mid \frac{m_i}{x_m(m_i)} \right]$ for $i = 1, \dots, s$ and $m_i \in \text{MinGen}(\text{in}_<(\mathfrak{a}))$. The rule (3.5) implies $t_i t_j \in \mathcal{B}$ if and only if $m(m_i) \geq m(m_j)$. It follows from Remark 3.4 that the set of fully attached i -tuples is in bijection with the set

$$\mathcal{B}_i := \left\{ e_{i_r} \dots e_{i_1} t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)} \mid \begin{array}{l} 1 \leq i_1 < \dots < i_r \leq n \\ 1 \leq j_1 < \dots < j_q \leq s \\ l_1, \dots, l_q \in \mathbb{N} \text{ and } i = r + 2 \sum_{t=1}^q l_t \end{array} \right\}.$$

For $f_j = m_j + \sum_{\alpha \in \mathbb{N}^n} f_{j\alpha} x^\alpha$ we define

$$T_p(f_j) := \sum_{\substack{\alpha \in \mathbb{N}^n \\ p = \max(\text{supp}(\alpha))}} f_{j\alpha} \frac{x^\alpha}{x_p}.$$

We have the following theorem:

Theorem 3.10. *Let $\mathfrak{a} = \langle f_1, \dots, f_s \rangle \trianglelefteq S$ be an ideal, such that $f_1, \dots, f_s$ is a reduced Gröbner basis with respect to the degree-lex order and such that the initial ideal $\text{in}_<(\mathfrak{a})$ is a complete intersection, and let $A := S/\mathfrak{a}$ be the quotient algebra. Then the following complex is a minimal A -free resolution of the residue class field k and carries the structure of a differential-graded algebra:*

$$\begin{aligned} F_i := & \bigoplus_{\substack{1 \leq i_1 < \dots < i_r \leq n \\ 1 \leq j_1 < \dots < j_q \leq s \\ l_1, \dots, l_q \in \mathbb{N} \\ i = r + 2 \sum_{j=1}^q l_j}} A e_{i_r} \dots e_{i_1} t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)} \\ e_{i_r} \dots e_{i_1} & \xrightarrow{\partial} \sum_{m=1}^r (-1)^{\#\{i_j > i_m\}} x_{i_m} e_{i_r} \dots \widehat{e_{i_m}} \dots e_{i_1} \\ t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)} & \xrightarrow{\partial} \sum_{m=1}^s \sum_{p=1}^n T_p(f_{j_m}) e_p t_{j_1}^{(l_1)} \dots t_{j_m}^{(l_{j_m}-1)} \dots t_{j_q}^{(l_q)}, \end{aligned}$$

where $t_{i_j}^{(0)} := 1$, $e_i e_j = -e_j e_i$, and $e_i e_i = 0$. The differential is given by

$$\begin{aligned} \partial(e_{i_r} \dots e_{i_1} t_{j_1}^{(l_1)} \dots t_{j_s}^{(l_s)}) &= \partial(e_{i_r} \dots e_{i_1}) t_{j_1}^{(l_1)} \dots t_{j_s}^{(l_s)} \\ &\quad + (-1)^r e_{i_r} \dots e_{i_1} \partial(t_{j_1}^{(l_1)} \dots t_{j_s}^{(l_s)}). \end{aligned}$$

In particular, we have

$$P_k^A(\underline{x}, t) = \frac{\prod_{i=1}^n (1 + x_i t)}{\prod_{i=1}^k (1 - m_i t^2)}.$$

PROOF. By Theorem 3.6 it follows that the described complex is a free resolution of k and that it is minimal. The fact that the resolution carries the structure of a differential graded algebra is also easily seen once the asserted form of the differential is verified. Thus it remains to calculate the differential ∂ :

Let $[w_1 | \dots | w_l]$ be a fully attached tuple such that w_j is either a variable or a minimal fully attached tuple.

First assume that w_j is a variable, i.e. $w_j = x_{r_j}$. We prove that $w_i w_j$ can be permuted to $w_j w_i$ for all $i \neq j$. If w_i is a variable, say $w_i = x_{j_i}$, we have by (3.5) $j_i > r_j$ it follows $|x_{j_i} x_{r_j}| \rightarrow |x_{j_i} x_{r_j}| \rightarrow |x_{r_j} x_{j_i}|$. If w_i is a minimal fully attached tuple, i.e. $w_i = \left| x_{m(m_i)} \left| \frac{m_i}{x_{m(m_i)}} \right| \right|$, we have

$$\begin{aligned} \left| x_{m(m_i)} \left| \frac{m_i}{x_{m(m_i)}} \right| x_{r_j} \right| &\rightarrow \left| x_{m(m_i)} \left| x_{r_j} \frac{m_i}{x_{m(m_i)}} \right| \right| \rightarrow \left| x_{m(m_i)} \left| x_{r_j} \left| \frac{m_i}{x_{m(m_i)}} \right| \right| \right| \\ &\rightarrow \left| x_{r_j} x_{m(m_i)} \left| \frac{m_i}{x_{m(m_i)}} \right| \right| \rightarrow \left| x_{r_j} \left| x_{m(m_i)} \left| \frac{m_i}{x_{m(m_i)}} \right| \right| \right|. \end{aligned}$$

In the first case we have a reduction with coefficient -1 and in the second case with coefficient $+1$. Therefore, it is enough to consider the number of w'_i 's, $i < j$, which are variables. It follows that w_j can be permuted to the left with coefficient $(-1)^{\#\{w_i \mid w_i \text{ variable and } w_i <_{\text{lex}} x_{r_j}\}}$.

Now let w_j be a minimal fully attached tuple, i.e. $w_j = \left| x_{m(m_j)} \left| \frac{m_j}{x_{m(m_j)}} \right| \right|$. Then we have

$$\left[x_{m(m_j)} \left| \frac{m_j}{x_{m(m_j)}} \right| \right] \rightarrow - \sum_{\alpha} f_{j\alpha} [x^{\alpha}] \rightarrow \sum_{\alpha} f_{j\alpha} \left[x_{\alpha} \left| \frac{x^{\alpha}}{x_{\alpha}} \right| \right],$$

where $x_{\alpha} := x_{m(x^{\alpha})}$. Since $\left[\frac{x^{\alpha}}{x_{\alpha}} \right]$ is matched with $\left[x_{\beta} \left| \frac{x^{\alpha}}{x_{\beta} x_{\alpha}} \right| \right]$ (where $x_{\beta} = x_{m(x^{\beta})}$ with $x^{\beta} := \frac{x^{\alpha}}{x_{\alpha}}$) the exponent α decreases successively up to the element $[x_p]$ with $p = \max(\text{supp}(\alpha))$. Therefore, we get

$$(3.6) \quad \left[x_{m(m_j)} \left| \frac{m_j}{x_{m(m_j)}} \right| \right] \rightarrow \sum_{p=1}^n T_p(f_j) e_p.$$

We now consider the tuple $[w_1 | \dots | w_l]$. With the same argument as before, one can check that the minimal fully attached tuple w_j can be permuted with coefficient $+1$ to the right. After a chain of reductions, we reach the tuple

$$[w_j | w_1 | \dots | w_{j-1} | w_{j+1} | \dots | w_l].$$

Applying Equation (3.6) we get

$$[w_1 | \dots | w_l] \rightarrow \sum_{p=1}^n T_p(f_j) [x_p | w_1 | \dots | \widehat{w_j} | \dots | w_l].$$

In order to reach a fully attached tuple we have to permute the variable x_p to the correct position. This permutation yields a coefficient

$$(-1)^{\#\{w_i \mid w_i \text{ variable and } w_i <_{\text{lex}} x_p\}}.$$

The bijection between the elements $e_{i_r} \dots e_{i_1} t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)}$ and the fully attached tuples finally implies the coefficient

$$\left((-1)^{\#\{w_i \mid w_i \text{ variable and } w_i <_{\text{lex}} x_p\}} \right)^2 (-1)^r = (-1)^r.$$

Therefore, our differential has the desired form

$$\begin{aligned}
& \partial(e_{i_r} \dots e_{i_1} t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)}) \\
&= \sum_{m=1}^r (-1)^{\#\{i_j > i_m\}} x_{i_m} e_{i_r} \dots \widehat{e_{i_m}} \dots e_{i_1} t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)} \\
&+ \sum_{m=1}^q \sum_{\substack{p=1 \\ p \neq i_1, \dots, i_r}}^n (-1)^r T_p(f_{j_m}) e_{i_r} \dots e_{i_1} e_p t_{j_1}^{(l_1)} \dots t_{j_m}^{(l_{j_m}-1)} \dots t_{j_q}^{(l_q)}.
\end{aligned}$$

It is easy to see that these are all possible reductions. $\square$

If $\text{in}_{\prec}(\mathbf{a}) = \mathbf{a}$, then the preceding result about the Poincaré-Betti series can be found in [22].

CHAPTER 4

Resolution of the Residue Field in the Non-Commutative Case

In this chapter we study the same setting as in Chapter 3 but we replace the commutative polynomial ring by the polynomial ring in n non-commuting indeterminates.

We begin by introducing basics on non-commutative Gröbner bases and their relations to algebraic discrete Morse theory in Section 1. In the non-commutative case, the acyclic matching on the normalized Bar resolution is slightly different than the acyclic matching from Chapter 3, and the resulting Morse complex will be isomorphic to the Anick resolution, which will be shown in Section 2. These results were independently obtained by Sköldberg [40]. In addition to Sköldberg's results, we prove minimality of this resolution in special cases. In Section 3 we will exploit this result and derive information about the Poincaré-Betti series. We also give an explicit description of the complex if the two-sided ideal $\mathfrak{a}$ admits a (finite) quadratic Gröbner basis. Finally, in Section 4 we prove a conjecture by Sturmfels [43] on non-commutative lattice ideals.

1. Non-commutative Gröbner Bases and Discrete Morse Theory

Let $A = k\langle x_1, \dots, x_n \rangle / \mathfrak{a}$ be the quotient algebra of the polynomial ring in n non-commuting indeterminates by a two-sided ideal

$$\mathfrak{a} \trianglelefteq k\langle x_1, \dots, x_n \rangle.$$

As before, we assume that $\mathfrak{a} = \langle f_i \mid i \in I \rangle$ such that $\{f_i \mid i \in I\}$ is a minimal reduced Gröbner basis with respect to a fixed degree-monomial order $\prec$. For an introduction to the theory of Gröbner bases in the non-commutative case, see [44], [45], [26]. Note that in this situation almost all Gröbner basis are infinite. Nevertheless, we have again for the product of any two standard monomials a unique (Gröbner) representation of the form:

$$w \cdot v := \sum_i a_i w_i \text{ with } a_i \in k, w_i \in \mathcal{SM}, \text{ and } |w \cdot v| \geq |w_i| \text{ for all } i,$$

where $\mathcal{SM}$ is the corresponding set of standard monomials of degree ≥ 1 and $|m|$ is the total degree of the monomial m .

Note that in the proofs of the previous chapter we did not use the fact that the Gröbner basis is finite. Therefore, we can proceed in the same way as in Chapter 3. The acyclic matching on the normalized Bar resolution is defined as follows: As in the commutative case, we define $\mathcal{M}_j$ by induction on the coordinate $1 \leq j \leq n$:

For $j = 1$ we set

$$\mathcal{M}_1 := \left\{ \begin{array}{c} [x_i | w'_1 | w_2 | \dots | w_l] \\ \downarrow \\ [w_1 | \dots | w_l] \end{array} \in G(\mathbf{NB}_\bullet^A) \mid w_1 = x_i w'_1 \right\}.$$

The critical cells with respect to $\mathcal{M}_1$ are given by

- (1) $\mathcal{B}_1^{\mathcal{M}_1} := \{[x_i] \mid 1 \leq i \leq n\}$ in homological degree 1.
- (2) For $l \geq 2$ the critical cells in homological degree l are given by the set $\mathcal{B}_l^{\mathcal{M}_1}$ of all $[x_i | w_2 | w_3 | \dots | w_l]$, $w_2, w_3, \dots, w_l \in \mathcal{SM}$, such that $x_i w_2$ is reducible.

Assume now $j \geq 2$ and $\mathcal{M}_{j-1}$ is defined. Let $\mathcal{B}^{\mathcal{M}_{j-1}}$ be the set of critical cells left after applying $\mathcal{M}_1 \cup \dots \cup \mathcal{M}_{j-1}$.

Let $\mathcal{E}_j$ denote the set of edges in $G(\mathbf{NB}_\bullet^A)$ that connect critical cells in $\mathcal{B}^{\mathcal{M}_{j-1}}$. The following condition on an edge in $\mathcal{E}_j$ will define the matching $\mathcal{M}_j$.

Definition 4.1 (Matching Condition). Let

$$\begin{array}{c} [x_{i_1} | w_2 | \dots | w_{j-1} | u_1 | u_2 | w_{j+1} | \dots | w_l] \\ \downarrow \\ [x_{i_1} | w_2 | \dots | w_{j-1} | w_j | w_{j+1} | \dots | w_l] \end{array}$$

be an edge in $\mathcal{E}_j$. In particular, $w_j = u_1 u_2$. We say that the edge satisfies the matching condition if

- (i) u_1 is a prefix of w_j ,
- (ii) $[x_{i_1} | w_2 | \dots | w_{j-1} | u_1 | u_2 | w_{j+1} | \dots | w_l] \in \mathcal{B}^{\mathcal{M}_{j-1}}$,
- (iii) $[x_{i_1} | w_2 | \dots | w_{j-1} | v_1 | v_2 | w_{j+1} | \dots | w_l] \notin \mathcal{B}^{\mathcal{M}_{j-1}}$ for each prefix v_1 of u_1 and $v_1 v_2 = w_j$.

$$\mathcal{M}_j := \left\{ \begin{array}{c} [x_{i_1} | w_2 | \dots | w_{j-1} | u_1 | u_2 | w_{j+1} | \dots | w_l] \\ \downarrow \\ [x_{i_1} | w_2 | \dots | w_{j-1} | w_j | w_{j+1} | \dots | w_l] \end{array} \in \mathcal{E}_j \text{ satisfying 4.1} \right\}.$$

The set of critical cells $\mathcal{B}_l^{\mathcal{M}_j}$ in homological degree $l \geq 1$ is given by

- (1) $\mathcal{B}_1^{\mathcal{M}_j} := \{[x_i] \mid 1 \leq i \leq n\}$,
- (2) $\mathcal{B}_2^{\mathcal{M}_j}$ consists of elements $[x_{i_1} | w_2]$ with $x_{i_1} w_2 \in \text{MinGen}(\text{in}_<(\mathfrak{a}))$,
- (3) $\mathcal{B}_l^{\mathcal{M}_j}$ consists of elements $[x_{i_1} | w_2 | w_3 | \dots | w_l] \in \mathcal{B}_l^{\mathcal{M}_{j-1}}$ such that for each prefix u of w_j we have $[x_{i_1} | w_2 | \dots | w_{j-1} | u | \dots | w_l] \notin \mathcal{B}_l^{\mathcal{M}_{j-1}}$ and $w_j w_{j+1}$ is reducible.

We finally set $\mathcal{M} := \bigcup_{j \geq 1} \mathcal{M}_j$ and we write $\mathcal{B}^{\mathcal{M}}$ for the set of critical cells with respect to $\mathcal{M}$.

With the same proof as for Lemme 3.2 in Chapter 3 we get the following analog of Lemma 3.2 in the non-commutative case.

Lemma 4.2. $\mathcal{M}$ defines an acyclic matching. □

2. The Anick Resolution

As in the commutative case, we give a second description of the Morse complex with respect to the acyclic matching from Lemma 4.2. In this case, this description shows that its Morse complex is isomorphic to the Anick resolution [1]. As

Definition 3.3 in the commutative case the following definition is inspired by a construction by Sturmfels, see Example 4.10.

Definition 4.3. Let $m_{i_1}, \dots, m_{i_{l-1}} \in \text{MinGen}(\text{in}_{\prec}(\mathfrak{a}))$ be monomials, such that for $j = 1, \dots, l-1$ we have $m_{i_j} = u_{i_j} v_{i_j} w_{i_j}$ with $u_{i_{j+1}} = w_{i_j}$ and $|u_{i_1}| = 1$. Then we call the l -tuple

$$[u_{i_1} | v_{i_1} w_{i_1} | v_{i_2} w_{i_2} | \dots | v_{i_{l-1}} w_{i_{l-1}}]$$

fully attached if for all $1 \leq i \leq l-2$ and each prefix u of $v_{i_{j+1}} w_{i_{j+1}}$ the monomial $v_{i_j} w_{i_j} u$ lies in $\mathcal{SM}$. We write $\mathcal{B}_j := \{[w_1 | \dots | w_j]\}$ for the set of fully attached j -tuples ($j \geq 2$) and $\mathcal{B}_1 := \{[x_1], \dots, [x_n]\}$.

We define the reduction types (Type I, Type II, and Type III) and the reduction coefficient $[e : f]$ for two fully attached tuples e, f analogous to the commutative case (see Definition 3.5). Now we are able to define the following complex:

$$F_j := \bigoplus_{e \in \mathcal{B}_j} A e,$$

$$\begin{aligned} \partial : F_i &\rightarrow F_{i-1} \\ e &\mapsto \sum_{f \in \mathcal{B}_{i-1}} [e : f] f. \end{aligned}$$

Note that the basis elements of F_j are exactly the basis elements in the Anick resolution (see [1]). This then implies that the complex $F_\bullet$ is isomorphic to the Anick resolution. Again, we have:

Theorem 4.4. $(F_\bullet, \partial)$ is an A -free resolution of the residue field k over A . If no reduction of Type II is possible then the resolution $(F_\bullet, \partial)$ is minimal.

PROOF. The fully attached tuples are exactly the critical cells. The rest is analogous to the commutative case. $\square$

If one applies Theorem 4.4 to the ideal $\mathfrak{a}'$ generated by $\mathfrak{a}$ together with the commutators $x_i x_j - x_j x_i$ for $1 \leq i < j \leq n$ then clearly the quotient $k\langle x_1, \dots, x_n \rangle / \mathfrak{a}'$ is a commutative algebra and is isomorphic to $k[x_1, \dots, x_n] / \mathfrak{a}''$, where $\mathfrak{a}''$ is the ideal generated in $k[x_1, \dots, x_n]$ by the elements of $\mathfrak{a}$, when seen as polynomials in commuting variables. But in general, the Morse complex with respect to the acyclic matching from Lemma 4.2 for the ideal $\mathfrak{a}'$ in $k\langle x_1, \dots, x_n \rangle$ is much larger (i.e., has higher ranks of the free modules in some homological degrees) than the Morse complex of the acyclic matching from Chapter 3 applied to the ideal $\mathfrak{a}''$ in $k[x_1, \dots, x_n]$.

Since coefficients $[e : f] \in k \setminus \{0\}$ can enter the resolution only by reductions of Type II, we have:

Proposition 4.5. *The following conditions are equivalent:*

- (1) $(F_\bullet, \partial)$ is not minimal.
- (2) *There exist standard monomials $w_1, \dots, w_4 \in \mathcal{SM}$ and minimal generators $m_{i_1}, m_{i_2}, m_{i_3} \in \text{MinGen}(\text{in}_{\prec}(\mathfrak{a}))$ such that $w_1 w_2 = u_1 m_{i_1}$, $w_2 w_3 = u_2 m_{i_2}$, $w_1 w_4 = u'_1 m_{i_3}$ with u_1, u'_1 suffixes of w_1 , u_2 suffix of w_2 , and $w_2 w_3 \rightarrow w_4 \in \mathcal{R}$*

PROOF. $(F_\bullet, \partial)$ is minimal if and only if no reduction of Type II is possible, which is equivalent to the second condition. $\square$

Corollary 4.6. *In the following two cases, the resolution $(F_\bullet, \partial)$ is a minimal A -free resolution of k and independent of the characteristic of k .*

- (1) $\mathfrak{a}$ admits a monomial Gröbner basis.
- (2) The Gröbner basis of $\mathfrak{a}$ consists of homogeneous polynomials, all of the same degree.

PROOF. If the Gröbner basis consists of monomials, the situation from Proposition 4.5 is not possible. In the other case, there exists a constant l such that for all $w \rightarrow v \in \mathcal{R}$ we have $|w| = |v| = l$. Assume there exist standard monomials $w_1, \dots, w_4$ and minimal generators $m_{i_1}, m_{i_2}, m_{i_3} \in \text{MinGen}(\text{in}_<(\mathfrak{a}))$ such that $w_1 w_2 = u_1 m_{i_1}$, $w_2 w_3 = u_2 m_{i_2}$, $w_1 w_4 = u'_1 m_{i_3}$ with u_1, u'_1 suffixes of w_1 , u_2 suffix of w_2 , and $w_2 w_3 \rightarrow w_4 \in \mathcal{R}$. Then we get $|w_i| < l$ for $i = 2, 3, 4$. On the other hand, we have $w_2 w_3 \rightarrow w_4 \in \mathcal{R}$ and therefore $|w_4| = l$. This is a contradiction. $\square$

Clearly, situation (1) from Corollary 4.6 means that the ideal $\mathfrak{a}$ is a monomial ideal.

3. The Poincaré-Betti Series of k

In this section we exploit Theorem 4.4 in order to draw corollaries on the Poincaré-Betti series of k . Assume that there exists a term order for which there is a finite reduced Gröbner basis. This assumption implies that the initial ideal $\text{in}_<\mathfrak{a}$ is finitely generated.

Recall that by Definition 3.3 an l -tuple $[u_{i_1} | v_{i_1} w_{i_1} | v_{i_2} w_{i_2} | \dots | v_{i_{l-1}} w_{i_{l-1}}]$ is called a fully attached l -tuple if there exist leading monomials $m_{i_1}, \dots, m_{i_{l-1}}$ from the set $\text{MinGen}(\text{in}_<(\mathfrak{a}))$ such that for all $j = 1, \dots, l-1$ there exist monomials $u_{i_j}, v_{i_j}, w_{i_j} \in \mathcal{SM}$ with $m_{i_j} = u_{i_j} v_{i_j} w_{i_j}$ and $u_{i_{j+1}} = w_{i_j}$. It follows that the fully attached l -tuples $[u_{i_1} | v_{i_1} w_{i_1} | v_{i_2} w_{i_2} | \dots | v_{i_{l-1}} w_{i_{l-1}}]$ are in one-to-one correspondence with $l-1$ chains of monomials $(m_{i_1}, \dots, m_{i_{l-1}})$ that satisfy the above conditions. We write again $\mathcal{B}$ for the set of all those chains. Now consider the subset

$$E := \left\{ (m_{i_1}, \dots, m_{i_l}) \in \mathcal{B} \mid m_{i_1}, \dots, m_{i_l} \text{ pairwise different} \right\} \subset \mathcal{B}.$$

Since we consider only finite Gröbner bases, it is clear that E is finite. We construct a DFA (deterministic finite automaton, see for example [25]) over the alphabet E , which accepts $\mathcal{B}$. For each letter $f \in E$ we define a state f . Each state f is a final state. Let S be the initial state and Q be an error state. From the state S we go to state f if we read the letter $f \in E$. Let $f_1 = (m_{i_1}, \dots, m_{i_l}), f_2 = (m'_{j_1}, \dots, m'_{j_{l'}}) \in E$ be two chains of monomials with corresponding fully attached tuples $[w_{i_1} | \dots | w_{i_{l+1}}]$ and $[w'_{j_1} | \dots | w'_{j_{l'+1}}]$. Then we have $(f_1, f_2) \in \mathcal{B}$ if and only if there exists a monomial $n \in \text{MinGen}(\text{in}_<(\mathfrak{a}))$ with $n = u w'_{j_1}$ and u suffix of $w_{i_{l+1}}$. In this case, we change by reading f_2 from state f_1 to f_2 . If such a monomial does not exist, we change by reading f_2 from state f_1 to the error state Q . The language of this DFA is exactly the set $\mathcal{B}$. This proves that the basis of our resolution $F_\bullet$ is a regular language. In particular, since the word-counting function of a regular language is always a rational function (see [25]), we get the following theorem:

Theorem 4.7. *For a ring A the word counting function $F(\underline{x}, t)$ of the set $\mathcal{B}$ is rational and*

$$P_k^A(\underline{x}, t) \leq F(\underline{x}, t).$$

Equality holds if and only if $F_\bullet$ is minimal. $\square$

Corollary 4.8. *For the following two cases the Poincaré-Betti series of k over the ring A is a rational function:*

- (1) $\mathfrak{a}$ admits a Gröbner basis consisting of monomials.
- (2) The Gröbner basis of $\mathfrak{a}$ consists of homogeneous polynomials, all of the same degree.

PROOF. The result is a direct consequence of the Theorem 4.7 and Corollary 4.6. $\square$

Moreover, we obtain a corollary the following well know result (see [39, Chapter 7]). We would like to mention that in general it is open whether the Hilbert-series of a Koszul algebra is rational (see also [39, Chapter 7]).

Corollary 4.9. *If $\mathfrak{a}$ has a quadratic Gröbner basis, then $F_\bullet$ is an A -free minimal linear resolution. Hence A is Koszul and its Hilbert and Poincaré-Betti series are rational functions.* $\square$

4. Examples

We finally want to give some examples, where an explicit description of the Morse complex is possible. In particular, we verify a conjecture by Sturmfels:

Example 4.10 (Conjecture by Sturmfels (see [43])). Let Λ be a graded sub-semigroup of $\mathbb{N}^d$ with n generators. That is $\Lambda = \mathbb{N}\lambda_1 + \cdots + \mathbb{N}\lambda_n$ for $\lambda_i \in \mathbb{N}^d$, $1 \leq i \leq n$, and there is a hyperplane in $\mathbb{N}^d$ that contains all λ_i . We write the (commutative) semigroup algebra $k[\Lambda]$ of Λ over a field k as a quotient of the polynomial ring in non-commuting variables

$$k\langle y_1, y_2, \dots, y_n \rangle / J_\Lambda = k[\Lambda].$$

Here, J_Λ is the ideal generated by all binomials $y_{i_1} \cdots y_{i_r} - y_{j_1} \cdots y_{j_s}$ for $1 \leq i_1, \dots, i_r, j_1, \dots, j_s \leq n$ for which $\lambda_{i_1} + \cdots + \lambda_{i_r} = \lambda_{j_1} + \cdots + \lambda_{j_s}$. In particular, J_Λ contains all commutation relations $y_i y_j - y_j y_i$ for $i < j \leq n$.

Suppose that the two-sided ideal J_Λ possesses a quadratic Gröbner basis $\mathfrak{G}$. The elements in the non-commutative Gröbner basis $\mathfrak{G}$ are quadratic reduction relations of the form $y_i y_j \rightarrow y_{i'} y_{j'}$. If w and w' are words in $y_1, \dots, y_n$, then we write $w \rightarrow_j w'$ if there exists a reduction sequence of length j from w to w' . Sturmfels calls a word $w = y_{i_1} y_{i_2} \cdots y_{i_r}$ *fully attached* if every quadratic subword $y_{i_j} y_{i_{j+1}}$ can be reduced with respect to $\mathfrak{G}$. It is easily seen that $y_{i_1} \cdots y_{i_r}$ is fully attached in the sense of Sturmfels if and only if $[y_{i_1} | \cdots | y_{i_r}]$ is a fully attached r -tuple in the sense of Definition 4.3.

Now the conjecture by Sturmfels, formulated in our terminology, states that if J_Λ has a quadratic Gröbner basis then there is a minimal free resolution for which the fully attached r -tuples comprise a basis of the r -th homological degree.

Let $\mathbf{F}_r$ be the free $k[\Lambda]$ -module with basis $\{E_w : w \text{ fully attached word of length } r\}$. Let $\mathbf{F} = \bigoplus_{r \geq 0} \mathbf{F}_r$ and define a differential ∂ on $\mathbf{F}$ as follows:

$$\partial : \mathbf{F}_r \rightarrow \mathbf{F}_{r-1}, \quad E_w \mapsto \sum (-1)^j x_i E_{w'},$$

where the sum is over all fully attached words w' of length $r-1$ such that $w \rightarrow_j x_i w'$ for some i, j . Note that this sum includes the trivial reduction $w \rightarrow_0 w$. Then

Theorem 4.4 together with Proposition 4.5 implies that $(\mathbf{F}, \partial)$ is a minimal free resolution of k over $k[\Lambda]$. $\square$

We give two examples for the conjecture by Sturmfels.

Example 1 (The twisted cubic curve): Let $\Lambda \subseteq \mathbb{N}^2$ be generated by

$$\lambda_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

Then the defining dependencies on the λ_i , $1 \leq i \leq 4$, are $\lambda_1 + \lambda_3 = \lambda_2 + \lambda_2$, $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3$, $\lambda_2 + \lambda_4 = \lambda_3 + \lambda_3$, and the obvious commutation relations.

Consider $k[\Lambda]$ as a quotient of the polynomial ring $k[a, b, c, d]$. here is a Gröbner basis consisting of nine binomials:

$$\mathfrak{G} = \left\{ \begin{array}{l} ac \rightarrow bb, ca \rightarrow bb, ad \rightarrow cb, da \rightarrow cb, bd \rightarrow cc, \\ db \rightarrow cc, ba \rightarrow ab, bc \rightarrow cb, dc \rightarrow cd \end{array} \right\}.$$

The minimal free resolution $(\mathbf{F}, \partial)$ has the format

$$\cdots \cdots \xrightarrow{\partial} k[\Lambda]^{72} \xrightarrow{\partial} k[\Lambda]^{36} \xrightarrow{\partial} k[\Lambda]^{18} \xrightarrow{\partial} k[\Lambda]^9 \xrightarrow{\partial} k[\Lambda]^4 \xrightarrow{\partial} k.$$

One of the 36 fully attached monomials of degree four is $adad$. It admits three relevant reductions $adad \rightarrow_0 adad$, $adad \rightarrow_1 cbad$ and $adad \rightarrow_3 bddb$. This implies

$$\partial(E_{adad}) = a \cdot E_{dad} - c \cdot E_{bad} - b \cdot E_{bdb}.$$

Example 2 (The Koszul complex): Let $\Lambda = \mathbb{N}^d$. There is a Gröbner basis $\mathfrak{G}$ consisting of the relations $y_i y_j \rightarrow y_j y_i$ for $1 \leq j < i \leq n$. A word w is fully attached if and only if $w = y_{i_1} y_{i_2} \cdots y_{i_r}$ for $i_1 > i_2 > \cdots > i_r$. In this case, $\partial(E_w) = \sum_{j=1}^r (-1)^{r-j} y_{i_j} E_{w_j}$ where $w_j = y_{i_1} \cdots y_{i_{j-1}} y_{i_{j+1}} \cdots y_{i_r}$. Hence $(\mathbf{F}, \partial)$ is the Koszul complex on n indeterminates.

Example 4.11 (The Cartan complex). If A is the exterior algebra, then $F_\bullet$ with

$$\begin{aligned} F_i &:= \bigoplus_{\substack{1 \leq j_1 < \cdots < j_r \\ l_1, \dots, l_r \in \mathbb{N} \\ i = \sum_{t=1}^r l_t}} A e_{i_1}^{(l_1)} \cdots e_{i_r}^{(l_r)} \\ e_{i_1}^{(l_1)} \cdots e_{i_r}^{(l_r)} &\rightarrow \sum_{t=1}^r x_{i_t} e_{i_1}^{(l_1)} \cdots e_{i_t}^{(l_t-1)} \cdots e_{i_r}^{(l_r)} \end{aligned}$$

defines a minimal free resolution of k as A -module, called the Cartan complex.

Proof. For the exterior algebra $A = k(x_1, \dots, x_n) / \langle x_i x_j + x_j x_i \rangle$ the resolution $F_\bullet$ is by Corollary 4.6 minimal. The set of reduction rules is given by $\mathcal{R} := \{x_i^2 \rightarrow 0, x_i x_j \xrightarrow{-1} x_j x_i \text{ for } i < j\}$. Then the fully attached tuples are exactly the words

$$(x_{i_1}, \dots, x_{i_1}, x_{i_2}, \dots, x_{i_2}, \dots, x_{i_r}, \dots, x_{i_r}) \text{ with } 1 \leq i_1 < \cdots < i_r \leq n.$$

Since $x_i x_j$ is reduced to $-x_j x_i$, if $i \neq j$, and each reduction has factor (-1) , we get for each reduction the coefficient $(-1)(-1) = 1$. Since $x_i x_i$ is reduced to 0, the differential follows. $\square$

The following example shows that even in the case where the Gröbner basis is not finite one can apply our theory:

Example 4.12. Consider the two-side ideal $\mathfrak{a} = \langle x^2 - xy \rangle$. By [26] there does not exist a finite Gröbner basis with respect to degree-lex for $\mathfrak{a}$. One can show that $\mathfrak{a} = \langle xy^n x - xy^{n+1} \mid n \in \mathbb{N} \rangle$ and that $\{xy^n x - xy^{n+1} \mid n \in \mathbb{N}\}$ is an infinite Gröbner basis with respect to degree-lex.

If one applies our matching from Lemma 4.2, it is easy to see that the critical cells are given by tuples of the form

$$[x|y^{n_1}|x|y^{n_2}|x|\dots|x|y^{n_l}|x] \text{ and } [x|y^{n_1}|x|y^{n_2}|x|\dots|x|y^{n_l}]$$

with $n_1, \dots, n_l \in \mathbb{N}$.

A degree argument implies that the Morse complex is even a minimal free resolution. Therefore, we get a minimal free resolution $F_\bullet$ of k over $A = k\langle x_1, \dots, x_n \rangle / \mathfrak{a}$.

In this case, this proves that k does not admit a linear resolution and hence A is not Koszul.

CHAPTER 5

Application to the Acyclic Hochschild Complex

Hochschild homology is defined for arbitrary associative algebras. There is an extensive literature on Hochschild homology and cohomology (see for example [10], [12]). Interest in explicit computations of the Hochschild homology is motivated by questions in the representation theory of algebras and their K-theory. Despite this interest there are surprisingly few cases of algebras A for which the Hochschild homology of A is known explicitly (see for example calculations in [27], [28], [29], [12], [41] or [31]).

In this chapter we provide in Section 2 new derivations for some of those calculation and give a description of Hochschild homology for wider and new classes of algebras A : In particular, we treat algebras for the initial ideal of its defining ideal is a complete intersection and the exterior algebra. Section 1 is devoted to the introduction of Hochschild homology and the description of the Morse matching that will be used for the application. The basic ideas will be similar to those applied in Chapters 3 and 4.

1. Hochschild Homology and Discrete Morse Theory

Since we only consider the case $A = k\langle x_1, \dots, x_n \rangle / \mathfrak{a}$, we restrict our definition of Hochschild homology to algebras of this type. For general definitions we refer to [10] and [47]. Our proofs still hold if A is an R -algebra, where R is a commutative Ring and A is free as an R -module.

Let $A = k\langle x_1, \dots, x_n \rangle / \mathfrak{a}$ be the quotient algebra of a non-commutative polynomial ring by a two-side ideal $\mathfrak{a}$. The Hochschild complex $\mathrm{HC}_\bullet^A = (HC_i, \partial_i)$ is defined as follows:

$$\begin{aligned} HC_i &:= A^{\otimes_k i+2}, \text{ for } i \leq -1, \\ \partial(a_0 \otimes \dots \otimes a_{i+1}) &:= \sum_{j=0}^i (-1)^j a_0 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_{i+1}. \end{aligned}$$

Here $A \otimes A^{\mathrm{op}}$ acts on HC_i via

$$(\mu \otimes \gamma)(a_0 \otimes \dots \otimes a_{i+1}) = (\mu a_0) \otimes a_1 \otimes \dots \otimes a_i \otimes (a_{i+1}) \gamma.$$

Let W be a basis of A as k -vector space such that $1 \in W$. Then we can fix a basis of HC_i as $(A \otimes_k A^{\mathrm{op}})$ -module:

$$\left\{ [w_1 | \dots | w_i], \text{ with } w_j \in W. \right\}$$

Here we identify $[w_1 | \dots | w_i]$ with $1 \otimes w_1 \otimes \dots \otimes w_i \otimes 1$.

In [10] it is proved that $\mathrm{HC}_\bullet^A$ is a projective resolution of A as an $A \otimes A^{\mathrm{op}}$ -module, called the standard Hochschild resolution or the acyclic Hochschild complex.

We now consider the normalized acyclic Hochschild complex:

The following complex is called normalized standard Hochschild resolution, which is homotopy equivalent to the standard Hochschild resolution (a proof for the homotopy equivalence via algebraic Morse theory is given in Appendix A):

- (1) $HC_i^{\mathcal{M}}$ has basis $\{[w_1 | \dots | w_i] \mid w_j \in W \setminus \{1\}\}$.
- (2) The differential is given by

$$\begin{aligned} \partial_i([w_1 | \dots | w_i]) &= (w_1 \otimes 1) [w_2 | \dots | w_i] + (-1)^i (1 \otimes w_i) [w_1 | \dots | w_{i-1}] \\ &\quad + \sum_{j=1}^{i-1} (-1)^j \left(\sum_l a_l [w_1 | \dots | w_{j-1} | w'_l | w_{j+2} | \dots | w_i] \right), \end{aligned}$$

$$\text{if } w_j w_{j+1} = a_0 + \sum_l a_l w'_l, \text{ with } a_0, a_l \in k \text{ and } w'_l \in W \setminus \{1\}.$$

If M is an A -bimodule, we regard it as a right $A \otimes A^{\text{op}}$ -module via $m(\mu \otimes \gamma) := \gamma m \mu$.

The Hochschild homology $HH(A, M)$ of A with coefficients in M is defined by the homology of the complex $M \otimes_{(A \otimes A^{\text{op}})} HC_{\bullet}^A$. In this case it follows that (see [10])

$$HH(A, M) \cong \text{Tor}_{\bullet}^{A \otimes A^{\text{op}}}(M, A).$$

Now let $A = k\langle x_1, \dots, x_n \rangle / \langle f_1, \dots, f_s \rangle$ (resp. $k[x_1, \dots, x_n] / \langle f_1, \dots, f_s \rangle$) be the non-commutative (resp. commutative) polynomial ring in n indeterminates divided by a two-sided ideal, where $f_1, \dots, f_s$ is a finite reduced Gröbner basis of $\mathfrak{a} = \langle f_1, \dots, f_s \rangle$ with respect to the degree-lex order. We now give an acyclic matching on the acyclic Hochschild complex, which is minimal in special cases. Let $\mathcal{SM}$ be the set of standard monomials of degree ≥ 1 with respect to the degree-lex order. In this case, the normalized acyclic Hochschild complex is given by

$$HC_i := \bigoplus_{w_1, \dots, w_i \in \mathcal{SM}} (A \otimes A^{\text{op}}) [w_1 | \dots | w_i]$$

with differential

$$\begin{aligned} \partial([w_1 | \dots | w_i]) &:= (w_1 \otimes 1) [w_2 | \dots | w_i] \\ &\quad + (-1)^i (1 \otimes w_i) [w_1 | \dots | w_{i-1}] \\ &\quad + \sum_{j=1}^{i-1} (-1)^j \left(\sum_r a_r [w_1 | \dots | w_{j-1} | v_r^j | w_{j+2} | \dots | w_i] \right) \end{aligned}$$

if $w_j w_{j+1}$ is reducible to $a_0 + \sum_r a_r v_r^j$ (if $w_j w_{j+1} \in \mathcal{SM}$, we set $v_r^j = w_j w_{j+1}$).

We adopt the notation from Chapter 4 and apply the same acyclic matching as in this chapter (resp. Chapter 3).

Since in addition in this case the differential maps the element $[w_1 | \dots | w_i]$ also to $(-1)^i (1 \otimes w_i) [w_1 | \dots | w_{i-1}]$ we have to modify the differential:

The reduction rules are the same as in Chapter 3, except that the reduction coefficient in Definition 3.5 is $(c \otimes 1)$ instead of c . In order to define the coefficient, we say e can be reduced to f with coefficient c (we write $e \xrightarrow{c} f$), where $e = (w_1, \dots, w_l)$ and $f = (v_1, \dots, v_{l-1})$ are two fully attached l (resp. $l-1$)-tuples, if there exists a sequence of l -tuples $e = e_0, e_1, \dots, e_{r-1}$ such that either there exists:

- (1) an l -tuple $e_r = (u, f)$ such that e_0 can be reduced to e_r with reductions of Type I and III, i.e.

$$e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r;$$

in this case we set $c := ((-1)^r \prod_{i=1}^r a_i) (u \otimes 1)$, or

- (2) an l -tuple $e_r = (f, u)$ such that e_0 can be reduced to e_r with reductions of Type I and III, i.e.

$$e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r;$$

in this case we set $c := ((-1)^{r+k} \prod_{i=1}^r a_i) (1 \otimes u)$, or

- (3) an l -tuple e_r such that e_0 can be reduced to e_r with reductions of Type I and III and e_r can be reduced to f with a reduction of Type II, i.e.

$$e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r \xrightarrow{(-1)^j b} f;$$

in this case we set $c := (-1)^{r+j} b \prod_{i=1}^r a_i$.

We define the reduction coefficient $[e : f]$ and the complex $F_\bullet$ as in Chapter 4 (resp. Chapter 3). With the same proof as in Chapter 4 (resp. Chapter 3) we obtain the following theorem:

Theorem 5.1. *$(F_\bullet, \partial)$ is a free resolution of A as an $A \otimes A^{\text{op}}$ -module. If no reduction of Type II is possible, then $(F_\bullet, \partial)$ is minimal.* $\square$

Moreover, we get similar results to the results from Chapter 3, 4 about minimality of $F_\bullet$ and rationality of the Poincaré-Betti series

$$P_k^{A \otimes A^{\text{op}}}(\underline{x}, t) = \sum_{i, \alpha} \dim_k \left((\text{Tor}_i^{(A \otimes A^{\text{op}})}(k, A))_\alpha \right) \underline{x}^\alpha t^i$$

from Chapter 4 (resp. Chapter 3).

2. Explicit Calculations of Hochschild Homology

We can give an explicit description of the minimal free resolution $F_\bullet$ in the following cases:

- (1) $A = S/\langle f_1, \dots, f_s \rangle$, where $S = k[x_1, \dots, x_n]$ is the commutative polynomial ring in n indeterminates and f_i a reduced Gröbner basis with respect to the degree-lex order such that the initial ideal is a complete intersection (note that in case $s = 1$ this resolution was first given by BACH (see [12])).
- (2) $A = E$, where E is the exterior algebra.

Note that the first class of algebras coincides with on the classes treated in Chapter 3.

Let $A = k[x_1, \dots, x_n]/\langle f_1, \dots, f_s \rangle$ be the commutative polynomial ring in n indeterminates with $f_i = x^{\gamma_i} + \sum_{\alpha_i \neq 0} f_{i, \alpha_i} x^{\alpha_i}$, $1 \leq i \leq s$, a reduced Gröbner basis with respect to the degree-lex order such that x^{γ_i} is the leading term (since we start with the normalized Hochschild resolution, the condition $\alpha \neq 0$ is no restriction).

Let $\mathcal{SM} = \{x^\alpha \mid x^\alpha \notin \langle x^{\gamma_1}, \dots, x^{\gamma_s} \rangle\}$ be the set of standard monomials of degree ≥ 1 . We assume that the initial ideal $\text{in}_<(\mathfrak{a}) = \langle x^{\gamma_1}, \dots, x^{\gamma_s} \rangle$ is a complete

intersection. With the same arguments as in Theorem 3.10 it follows that $F_\bullet$ is minimal. We use the same notation as [12] and write

$$\begin{aligned} T(x_i) &= (x_i \otimes 1) - (1 \otimes x_i), \\ \frac{T_i(f)}{T(x_i)} &= \sum_{\alpha \in \mathbb{N}^n} f_\alpha \sum_{j=0}^{\alpha_i-1} (x^{\alpha_1} \dots x^{\alpha_{i-1}} x^j \otimes x^{\alpha_i-1-j} x^{\alpha_{i+1}} \dots x^{\alpha_n}). \end{aligned}$$

Under these conditions, we get the following result:

Theorem 5.2. *Let $A = S/\langle f_1, \dots, f_s \rangle$ such that the initial ideal*

$$\text{in}_<(\langle f_1, \dots, f_s \rangle)$$

is a complete intersection. Then the following complex is a multigraded minimal resolution of A as an $A \otimes A^{\text{op}}$ -module and carries the structure of a differential-graded algebra:

$$\begin{aligned} F_i := & \bigoplus_{\substack{1 \leq i_1 < \dots < i_r \leq n \\ 1 \leq j_1 < \dots < j_q \leq s \\ l_1, \dots, l_q \in \mathbb{N} \\ i = r + 2 \sum_{j=1}^q l_j}} A \otimes A^{\text{op}} e_{i_r} \dots e_{i_1} t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)}, \end{aligned}$$

$$e_{i_r} \dots e_{i_1} \xrightarrow{\partial} \sum_{m=1}^r (-1)^{\#\{i_j > i_m\}} T(x_{i_m}) e_{i_r} \dots \widehat{e_{i_m}} \dots e_{i_1},$$

$$t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)} \xrightarrow{\partial} \sum_{m=1}^q \sum_{p=1}^n \frac{T_p(f_{j_m})}{T(x_p)} e_p t_{j_1}^{(l_1)} \dots t_{j_m}^{(l_{j_m}-1)} \dots t_{j_q}^{(l_q)},$$

where $t_{i_j}^{(0)} := 1$, $e_i e_j = -e_j e_i$, and $e_i e_i = 0$. For the differential we have:

$$\begin{aligned} \partial(e_{i_r} \dots e_{i_1} t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)}) &= \partial(e_{i_r} \dots e_{i_1}) t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)} \\ &\quad + (-1)^r e_{i_r} \dots e_{i_1} \partial(t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)}). \end{aligned}$$

Note that in case $A = S/\langle f \rangle$ this result was first obtained in [12] and our complex coincides with the complex given in [12].

Corollary 5.3. *Under the assumptions of Theorem 5.2 the Hilbert series of the Hochschild homology of A with coefficients in k has the form:*

$$\begin{aligned} \text{Hilb}_{HH(A,k)}(\underline{x}, t) &= \sum_{i, \alpha} \dim_k \left((\text{Tor}_i^{A \otimes A^{\text{op}}}(k, A)_\alpha) x^\alpha t^i \right) \\ &= \frac{\prod_{i=1}^n (1 + x_i t)}{\prod_{i=1}^k (1 - \underline{x}^{\gamma_i} t^2)}. \end{aligned}$$

If $\mathfrak{a}$ is the zero-ideal, we get with the same arguments the following special case:

Corollary 5.4. *Let $A = k[x_1, \dots, x_n]$, then the following complex is a minimal free resolution of A as an $A \otimes A^{\text{op}}$ -module:*

$$F_i := \bigoplus_{1 \leq i_1 < \dots < i_r \leq n} A \otimes A^{\text{op}} e_{i_1} \dots e_{i_r},$$

$$e_{i_1} \dots e_{i_r} \xrightarrow{\partial} \sum_{m=1}^r (-1)^{\#\{i_j < i_m\}} T(x_{i_m}) e_{i_1} \dots \widehat{e_{i_m}} \dots e_{i_r}.$$

In particular, we have:

$$\begin{aligned} \text{Hilb}_{HH(A,k)}(\underline{x}, t) &= \sum_{i, \alpha} \dim_k \left((\text{Tor}_i^{A \otimes A^{\text{op}}})(k, A)_\alpha \right) \underline{x}^\alpha t^i \\ &= \prod_{i=1}^n (1 + x_i t). \end{aligned}$$

PROOF OF THEOREM 5.2. The description of the basis of F_i follows with exactly the same arguments as for the proof of Theorem 3.10. Since no constant term appears in the differential it suffices, to verify that the differential has the given form.

First, we consider a variable $[x_i]$. Clearly, it maps to $(x_i \otimes 1) - (1 \otimes x_i)$.

Next, we consider a minimal fully attached tuple $w_j = \left[x_{\gamma_i} \left| \frac{x^{\gamma_i}}{x_{\gamma_i}} \right. \right]$, where $x_\gamma := x_{m(x^\gamma)}$. Then we have:

$$\left[x_{\gamma_i} \left| \frac{x^{\gamma_i}}{x_{\gamma_i}} \right. \right] \rightarrow - \sum_{\alpha} f_{i\alpha} [x^\alpha] \rightarrow \sum_{\alpha} f_{i\alpha} \left[x_\alpha \left| \frac{x^\alpha}{x_\alpha} \right. \right].$$

As in the commutative case, the multi-index α decreases successively, but here $\left[x_\beta \left| \frac{x^\alpha}{x_\beta x_\alpha} \right. \right]$, for $x_\beta = x_{m(x^\beta)}$ with $x^\beta := \frac{x^\alpha}{x_\alpha}$, maps in addition to $\left(1 \otimes \frac{x^\alpha}{x_{a'} x_\alpha} \right) [x_{a'}]$, hence in this case we get:

$$\left[x_{\gamma_i} \left| \frac{x^{\gamma_i}}{x_{\gamma_i}} \right. \right] \rightarrow \sum_{j=1}^n \frac{T_j(f_i)}{T(x_j)} e_j.$$

For a fully attached tuple $[w_1 | \dots | w_l]$, we have to calculate the sign of the corresponding permutations. This calculation is similar to the calculation of the sign in the commutative case (see proof of Theorem 3.10) and is left to the reader.

With the bijection between the elements $e_{i_r} \dots e_{i_1} t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)}$ and the fully attached tuples, we finally get the following differential:

$$\begin{aligned} &\partial(e_{i_r} \dots e_{i_1} t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)}) \\ &= \sum_{m=1}^r (-1)^{\#\{i_j > i_m\}} T(x_{i_m}) e_{i_r} \dots \widehat{e_{i_m}} \dots e_{i_1} t_{j_1}^{(l_1)} \dots t_{j_q}^{(l_q)} \\ &\quad + \sum_{m=1}^q \sum_{\substack{p=1 \\ p \neq i_1, \dots, i_r}}^n (-1)^r \frac{T_p(f_{j_m})}{T(x_p)} e_{i_r} \dots e_{i_1} e_p t_{j_1}^{(l_1)} \dots t_{j_m}^{(l_{j_m}-1)} \dots t_{j_q}^{(l_q)}, \end{aligned}$$

and the desired result follows. $\square$

We now consider the exterior algebra:

Theorem 5.5. *Let $E = k[x_1, \dots, x_n] / \langle x_i^2, x_i x_j + x_j x_i \rangle$ be the exterior algebra. The following complex is a minimal free resolution of E as $E \otimes E^{\text{op}}$ -module:*

$$\begin{aligned} F_i := & \bigoplus_{\substack{1 \leq i_1 < \dots < i_r \leq n \\ l_1, \dots, l_r \in \mathbb{N}^n}} E \otimes E^{\text{op}} e_{i_1}^{(l_1)} \dots e_{i_r}^{(l_r)} \end{aligned}$$

with

$$e_{i_1}^{(l_1)} \dots e_{i_r}^{(l_r)} \sum_{j=1}^{\mapsto r} (x_{i_j} \otimes 1) + (1 \otimes x_{i_j}) e_{i_1}^{(l_1)} \dots e_{i_j}^{(l_j-1)} \dots e_{i_r}^{(l_r)}.$$

In particular, we have :

$$\begin{aligned} \text{Hilb}_{HH(E,k)}(\underline{x}, t) &= \sum_{i, \alpha} \dim_k ((\text{Tor}_i^{E \otimes E}(k, E))_\alpha) \underline{x}^\alpha t^i \\ &= \prod_{i=1}^n \frac{1}{1 - x_i t}. \end{aligned}$$

Let S be the commutative polynomial ring in n indeterminates, then we have the following duality:

$$\begin{aligned} \text{Hilb}_{HH(E,k)}(\underline{x}, t) &= \text{Hilb}_S(\underline{x}, t), \\ \text{Hilb}_{HH(S,k)}(\underline{x}, t) &= \text{Hilb}_E(\underline{x}, t). \end{aligned}$$

PROOF. Essentially, the proof is identical to the proof of Example 4.11 from Chapter 4. The only fact that has to be checked is that all arguments work for the slightly different differential from the Hochschild resolution. $\square$

CHAPTER 6

Minimal (Cellular) Resolutions for (p-)Borel Fixed Ideals

In this chapter we develop minimal free resolutions for principal Borel and a class of p -Borel fixed ideals. The resolutions are derived from the Taylor resolution by a single acyclic matching. Therefore, they are cellular resolutions in the sense of [8]. In Section 1 we define this concept and relate it to algebraic discrete Morse theory. In Section 2 we introduce Borel fixed ideals and describe a new cellular minimal free resolution of principle Borel fixed ideals. In Section 3 we then introduce p -Borel fixed ideals and give a cellular minimal free resolution for a new class of these ideals. In addition, we study their regularity and provide a conjecture that bounds their regularity from below.

1. Cellular Resolutions

Here we give a very short introduction to the concept of a cellular resolution. This concept appeared first in the work of Peeva, Bayer and Sturmfels [9] and Bayer and Sturmfels [8]. In [6] (topological) discrete Morse theory was first used to minimize cellular resolutions.

Let X be a CW-complex and $X_* = \cup_i X_*^{(i)}$ be the set of its open cells ($X_*^{(i)}$ is the set of cells of dimension i). On this set we define a partial order by $\sigma \prec \tau$, for $\sigma, \tau \in X_*$, if for the topological cells σ, τ we have that σ lies in the closure of τ ($\sigma \subset \bar{\tau}$). If $(P, \prec)$ is a partially ordered set and $\text{gr} : X_* \rightarrow P$ an order-preserving map, we call the tuple (X, gr) a P -graded CW-complex.

The homology $H_\bullet(X, R)$ of the CW-complex X with coefficients in R is defined as the homology of the cellular chain complex $C_\bullet(X)$, where the modules C_i are the free R -modules generated by the cells of dimension i , and the differential is given by

$$\partial(e_\sigma) = \sum_{\tau \subset \bar{\sigma}} [\sigma : \tau] e_\tau,$$

where $[\sigma : \tau]$, for an i -cell σ , is the topological degree of the map

$$S^{i-1} \xrightarrow{f_{\partial\sigma}} X^{(i-1)} \xrightarrow{\pi_\tau} S^{i-1}.$$

Here S^{i-1} denotes the $(i-1)$ -sphere, $X^{(i-1)}$ the $(i-1)$ -skeleton of X , $f_{\partial\sigma}$ the characteristic map of the attachment of the cell σ , and π_τ the canonical projection collapsing $X^{(i-1)} \setminus \tau$ to a point. For more details of the homology of CW-complexes see [48].

Definition 6.1. A $\mathbb{Z}^n$ -graded R -free resolution $C_\bullet$ of a $\mathbb{Z}^n$ -graded S -module M is called a cellular resolution if there exists a $\mathbb{Z}^n$ -graded CW-complex (X, gr) such that

- (1) for each $\alpha \in \mathbb{Z}^n, i \geq 0$, there exists a basis $\{c_\sigma | \sigma \in X_*^{(i)} \text{ and } \text{gr}(\sigma) = \alpha\}$ of $(C_i)_\alpha$,
- (2) for each $\sigma \in X_*^{(i)}$ we have:

$$\partial_i(c_\sigma) = \sum_{\sigma \succ \tau \in X_*^{(i-1)}} [\sigma : \tau] x^{\text{gr}(\sigma) - \text{gr}(\tau)} c_\tau,$$

where $[\sigma : \tau]$ is the coefficient in the cellular chain complex.

On cellular resolutions algebraic discrete Morse theory describes the effect of the discrete Morse theory developed by Forman [17] on the cellular chain complex. Since Forman's theory requires a regular CW-complex all edges selected by a matching from (topological) discrete Morse theory are also suitable for algebraic discrete Morse theory. Moreover, the resolution resulting for the application of algebraic discrete Morse theory is in this situation still supported by a CW-complex (see also [6]).

The additional geometric structure present in a cellular resolution makes minimal cellular resolutions particularly interesting. Assume that we know for a class of ideals that their minimal free resolution is cellular. Then the question of classifying the possible Betti-sequences for this class of ideals can be approached by studying the possible f -vectors of the CW-complexes supporting the minimal free resolution. Recall, the f -vector of a CW-complex X of dimension D is the vector $f^X = (f_0, \dots, f_d)$ where f_i is the number of i -cells used in the construction of X . Since questions about bounds on or the classification of Betti numbers of ideals and f -vectors of complexes of a given topological type are usually equally hard, instances where a correspondence between Betti numbers and f -vectors are known open new doors for attacking these questions.

In a series of papers (nearly Scarf ideals [37], poset ideal [38], [5] principal Cohen-Macaulay p-Borel fixed [5], matroid and oriented matroid ideals [33], [34] and [6]. essentially all classes of monomial ideals for which the minimal free resolution was known had been shown to have a minimal cellular resolution. Moreover, the results of this chapter adds more such classes. Therefore, for a while it was a challenging and interesting question whether or not all minimal free resolutions of monomial ideals can be realized as cellular resolutions. In [46] Velasco shows that this question has a negative answer. Recently, in his he not only provides a counterexample, but also gives conceptual reasons why the answer is negative. The proof is an intriguing combination of arguments from commutative algebra, algebraic topology and geometric group theory. The counterexample, actually shows that there are monomial ideals for which independent of the field there is no CW-complex supporting a minimal free resolution.

Note, that the resolutions, which we construct in this chapter are all cellular, since we apply algebraic discrete Morse theory to a resolution, supported by a regular CW-complex and hence all selected edges have weight ± 1 .

2. Cellular Minimal Resolution for Principal Borel Fixed Ideals

Throughout this section let $S := k[x_1, \dots, x_n]$ be the commutative polynomial ring in n variables and $\mathfrak{a} \trianglelefteq S$ a monomial ideal. Here we develop a minimal cellular resolution for principal Borel fixed ideals. A minimal free resolution and the regularity is known for general Borel fixed ideals (see for example [15]). Even a cellular minimal free resolution is known (the Lyubeznik resolution is in this case

a minimal cellular resolution [6]). We give another minimal cellular resolution for this type of ideals, which is a generalization of the *hypersimplex resolution of powers of the maximal homogeneous ideal* developed by Batzies and Welker [6].

Definition 6.2. A monomial ideal $\mathfrak{a} \trianglelefteq S$ is called Borel fixed, or stable, if for any monomial $m \in \mathfrak{a}$ and any $x_i \mid m$ we have

$$(6.1) \quad S_{ji}(m) = \frac{x_j}{x_i} m \in \mathfrak{a} \text{ for all } j < i.$$

The interest in Borel fixed ideal is based in the fact that generic initial ideals with respect to the rev-lex term order are Borel fixed if k has characteristic 0. More precisely, for an ideal I in S its generic initial ideal $\text{gin}(I)$ is given as $\text{in}_{\prec}(AI)$ for a generic matrix A . Here we denote by AI the ideal of all Af for $f \in A$, where Af is the polynomial which arises from f by the linear transformation for the variables which is given by A . Recall, (see [14, Theorem 15.18]) that for an ideal I in S there is Zariski open subset of the set of invertible $n \times n$ matrices for which $J = \text{in}_{\prec}(AI)$ is constant. If the characteristic of k is 0 then J is Borel fixed. Borel fixed ideals have been subject to intensive study in commutative algebra (see for example [7], [15], [21], [24], [36], etc.)

Let us summarize a few well known facts about Borel fixed ideals.

Remark 6.3. For a Borel fixed ideal $\mathfrak{a} \trianglelefteq S$ there exists a unique set $\mathfrak{B} := \{m_1, \dots, m_l\}$ of monomials such that $\mathfrak{B}$ is a minimal Borel-generating system of the ideal $\mathfrak{a}$, in the sense that any monomial $m \in \mathfrak{a}$ has a decomposition $m = u v$ such that v is constructed from a monomial m_r in $\mathfrak{B}$ by iterative applications of rule (6.1), i.e. there exist indices $i_1, \dots, i_l$ and $j_1, \dots, j_l$ with $j_k < i_k$ for $k = 1, \dots, l$ such that $v = S_{j_1 i_1}(\dots(S_{j_l i_l}(m_r))\dots)$. The minimality of $\mathfrak{B}$ requires that the ideal $\mathfrak{a}'$ Borel-generated by a proper subset $\mathfrak{B}'$ of $\mathfrak{B}$ is a proper subideal of $\mathfrak{a}$. We call $\mathcal{SM}$ the Borel-generating system of $\mathfrak{a}$.

Definition 6.4. A Borel fixed ideal $\mathfrak{a} \trianglelefteq S$ is called principal Borel fixed if the Borel-generating system consists of one monomial, i.e. $\mathfrak{B} = \{m\}$ for $m \in S$.

Lemma 6.5. Let $\mathfrak{a} \trianglelefteq S$ be a principal Borel fixed ideal with $\mathfrak{B} := \{x_1^{i_1} \dots x_n^{i_n}\}$. Then

$$\mathfrak{a} = \prod_{j=1}^n I_j$$

where

$$I_j := \begin{cases} \langle x_1, \dots, x_j \rangle^{i_j} & , \quad i_j > 0 \\ 1 & , \quad i_j = 0. \end{cases}$$

PROOF. The lemma is a direct consequence of Remark 6.3. $\square$

We now give a generalization of the hypersimplex resolution introduced by Batzies and Welker [6]. This resolution will be used to construct a minimal cellular resolution for principal Borel fixed ideals.

We first recall the definition of a hypersimplicial complex from [6]:

Definition 6.6 (see [6]). Let C_d^n be the polytopal CW-complex with

$$\Delta_n := n \cdot \Delta^{d-1} = \{(y_1, \dots, y_d) \in \mathbb{R}^d \mid \sum_{i=1}^d y_i = n, y_i \geq 0, i = 1, \dots, d\}$$

as underlying space and CW-complex structure induced by intersection with the cubical CW-complex structure on $\mathbb{R}^d$ given by the integer lattice $\mathbb{Z}^d$. That is the closed cells of C_d^n are given by all hypersimplices

$$C_{\alpha,J} := \text{conv} \left(\alpha + \sum_{j \in J} \varepsilon_j e_j \mid \varepsilon_j \in \{0,1\}, \sum_{j \in J} \varepsilon_j = n - |\alpha| \right)$$

with $\alpha \in \mathbb{N}^d, J \subset \{1, \dots, d\}, |\alpha| = \sum_{i=1}^d \alpha_i e_i$ the i th unit vector in $\mathbb{R}^d$, either subject to the conditions $|\alpha| = n$ and $J = \emptyset$ or the condition $1 \leq n - |\alpha| \leq |J| - 1$. The CW-complex is multigraded by setting $\text{lcm}(C_{\alpha,J}) := \alpha + \sum_{j \in J} e_j$. We refer to C_d^n as the hypersimplicial complex.

Example 6.7. Consider the ideal $\langle x_1, x_2 \rangle^2 = \langle x_1^2, x_1 x_2, x_2^2 \rangle$. The hypersimplicial complex consists of the following cells:

$$\begin{array}{ll} C_{(2,0),\emptyset} & |\alpha| = 2, J = \emptyset \\ C_{(1,1),\emptyset} & |\alpha| = 2, J = \emptyset \\ C_{(0,2),\emptyset} & |\alpha| = 2, J = \emptyset \\ C_{(1,0),\{1,2\}} & 1 \leq 2 - |\alpha| \leq |J| - 1 \\ C_{(0,1),\{1,2\}} & 1 \leq 2 - |\alpha| \leq |J| - 1 \end{array}$$

It is easy to see that in this case the hypersimplicial complex defines a minimal free resolution of the ideal $\langle x_1, x_2 \rangle^2$.

The unit vectors $e_i, i = 1, \dots, n$, of R^n induce an orientation on the hypersimplices by setting $e_1 < \dots < e_n$. We call this orientation the canonical orientation.

Lemma 6.8 (see Section 5 in [6]). *Considering canonical orientations of these hypersimplices and denoting $J = \{j_0 < \dots < j_r\}$, $J_\nu := J \setminus \{j_\nu\}$ the differential of C_d^n is given by*

$$\begin{aligned} \partial C_{\alpha,J} &:= \sum_{\nu=0}^r (-1)^\nu (C_{\alpha,J_\nu} - C_{\alpha+e_{j_\nu},J_\nu}) & \text{if } 2 \leq n - |\alpha| \leq |J| - 2, \\ \partial C_{\alpha,J} &:= \sum_{\nu=0}^r (-1)^\nu C_{\alpha,J_\nu} & \text{if } 1 = n - |\alpha| \leq |J| - 2, \\ \partial C_{\alpha,J} &:= \sum_{\nu=0}^r (-1)^{\nu+1} C_{\alpha+e_{j_\nu},J_\nu} & \text{if } 2 \leq n - |\alpha| = |J| - 1, \\ \partial C_{\alpha,\{j_0,j_1\}} &:= C_{\alpha+e_{j_1},\emptyset} - C_{\alpha+e_{j_0},\emptyset} & \text{if } 1 = n - |\alpha|, \\ \partial C_{\alpha,\emptyset} &= 0 & \text{if } |\alpha| = n. \end{aligned}$$

Proposition 6.9 (see Fact 5.1 [6]). C_d^n defines a multigraded cellular free resolution of $\langle x_1, \dots, x_d \rangle^n$. $\square$

Now consider the Borel fixed ideal

$$\mathfrak{a} := \langle x_1, \dots, x_{d_1} \rangle^{a_1} \cdots \langle x_1, \dots, x_{d_k} \rangle^{a_k} \subseteq S$$

with $d_1 < \dots < d_k$ and $a_j \in \mathbb{N} \setminus \{0\}$. We set for all $j = 1, \dots, d_k$

$$\beta_j := \sum_{\substack{l=1 \\ d_l \geq j}}^k a_l.$$

We consider the subcomplex of the hypersimplicial complex $C_{d_k}^{\beta_1} = C_{d_k}^{a_1 + \dots + a_k}$ consisting of the hypersimplices $C_{\alpha,J}$ satisfying one of the following conditions:

- (1) $|\alpha| = a_1 + \dots + a_k$, $J = \emptyset$, and $\sum_{j=i}^{d_k} \alpha_j \leq \beta_i$ for all $i = 1, \dots, d_k$,
- (2) $1 \leq a_1 + \dots + a_k - |\alpha| \leq |J| - 1$ and for $i = 1, \dots, d_k$ we have
- $$\sum_{j=i}^{d_k} \gamma_j \leq \beta_i$$

where $\gamma = \alpha + \sum_{j \in J} e_j$.

We denote this subcomplex with $SC^{\beta\bullet}$.

Lemma 6.10. *The complex $SC^{\beta\bullet}$ with the differential inherited from the hypersimplicial complex $C_{d_k}^{a_1+\dots+a_k}$ is a subcomplex of $C_{d_k}^{a_1+\dots+a_k}$.*

PROOF. It is easy to see that if $C_{\alpha,J}$ is a hypersimplex satisfying Condition (1) or Condition (2), then any face of $C_{\alpha,J}$ satisfies one of the Conditions (1), (2). $\square$

We want to prove that the subcomplex $SC^{\beta\bullet}$ defines a cellular resolution for the ideal $\mathfrak{a} := \langle x_1, \dots, x_{d_1} \rangle^{a_1} \dots \langle x_1, \dots, x_{d_k} \rangle^{a_k} \subseteq S$. In order to do so, we define the following intersection of halfspaces

$$H_{\leq \beta} := \left\{ (y_1, \dots, y_{d_k}) \in \mathbb{R}^{d_k} \mid \sum_{j=i}^{d_k} y_j \leq \beta_i, i = 1, \dots, d_k \right\}.$$

Clearly, every hypersimplex $C_{\alpha,J}$ of the subcomplex $SC^{\beta\bullet}$ lies in $H_{\leq \beta}$. Furthermore, it is easy to see that no other hypersimplex $C_{\alpha,J}$ of the complex $C_{d_k}^{\beta_1}$ lies in $H_{\leq \beta}$. Therefore, we have

$$SC^{\beta\bullet} = C_{d_k}^{\beta_1} \cap H_{\leq \beta}.$$

We are now in position to prove the following proposition:

Proposition 6.11. *The subcomplex $SC^{\beta\bullet}$ defines a multigraded cellular free resolution of*

$$\mathfrak{a} = \langle x_1, \dots, x_{d_1} \rangle^{a_1} \langle x_1, \dots, x_{d_2} \rangle^{a_2} \dots \langle x_1, \dots, x_{d_k} \rangle^{a_k}.$$

PROOF. First we calculate the zero-cells. These cells are given by

$$C_{\alpha, \emptyset} = \{\alpha\}$$

satisfying $\sum_{j=i}^{d_k} \alpha_j \leq \beta_i$ for all $i = 1, \dots, d_k$. Clearly, the vectors α satisfying these

properties are in one-to-one correspondence with the minimal generators of $\mathfrak{a}$.

Using a result of Bayer and Sturmfels [8] (see Proposition 2.2.3 of [6] for its formulation that fits our setting here), $SC^{\beta\bullet} = C_{d_k}^{\beta_1} \cap H_{\leq \beta}$ supports a cellular resolution of $\mathfrak{a}$ if and only if $H_i((C_{d_k}^{\beta_1} \cap H_{\leq \beta})_{\leq \gamma}, k) = 0$ for all $i \geq 1$ and $\gamma \in \mathbb{R}^{d_k}$.

Note that

$$(C_{d_k}^{\beta_1})_{\leq \gamma} = \Delta_n \cap \{(y_1, \dots, y_{d_k}) \in \mathbb{R}^{d_k} \mid y_i \leq \gamma_i, i = 1, \dots, d_k\}$$

is contractible or empty for all $\gamma \in \mathbb{Z}^{d_k}$ (see [6]). Furthermore, $(C_{d_k}^{\beta_1})_{\leq \gamma}$ is convex. Therefore, the intersection of $(C_{d_k}^{\beta_1})_{\leq \gamma}$ with the convex set $H_{\leq \beta}$ does not change homology. It follows

$$H_i((SC^{\beta\bullet})_{\leq \gamma}) = 0 \text{ for all } \gamma \in \mathbb{R}^{d_k}.$$

Proposition 2.2.3 of [6] implies the assertion. $\square$

Using Forman's theory, In [6] the authors define a Morse-matching for the complex C_d^n such that the resulting Morse complex is a minimal cellular multigraded resolution of $\langle x_1, \dots, x_d \rangle^n$ (see Fact 5.2 of [6]). The acyclic matching $\mathcal{M}$ given in [6] is defined by

$$(6.2) \quad C_{\alpha, J} \rightarrow C_{\alpha + e_{\max J, J \setminus \{\max J\}}} \in \mathcal{M}$$

for all $\alpha \in \mathbb{N}^{d_\kappa}$, $J \subset [d_\kappa]$, such that $2 \leq n - |\alpha| \leq |J| - 1$ and $\max J \geq \max \alpha := \max\{i \in [d_\kappa] \mid \alpha_i \neq 0\}$.

From the definition of $SC^{\beta\bullet}$ it follows that if $C_{\alpha, J}$ (resp. $C_{\alpha + e_{\max J, J \setminus \{\max J\}}}$) lies in $SC^{\beta\bullet}$, then $C_{\alpha + e_{\max J, J \setminus \{\max J\}}}$ (resp. $C_{\alpha, J}$) lies in $SC^{\beta\bullet}$. Therefore, the matching $\mathcal{M}$ described by (6.2) is a well defined acyclic matching for our subcomplex $SC^{\beta\bullet}$.

Denote with $\tilde{C}_{d_k}^{\beta_1}$ (resp. $\widetilde{SC}^{\beta\bullet}$) the resulting Morse complex. With the same proof as for Fact 5.2 6.11 of [6] we get the following theorem:

Theorem 6.12. *The complex $\widetilde{SC}^{\beta\bullet}$ defines a minimal multigraded cellular free resolution of*

$$\mathfrak{a} = \langle x_1, \dots, x_{d_1} \rangle^{a_1} \langle x_1, \dots, x_{d_2} \rangle^{a_2} \cdots \langle x_1, \dots, x_{d_k} \rangle^{a_k}.$$

3. Cellular Minimal Resolution for a Class of p -Borel Fixed Ideals

In this section we develop a minimal cellular resolution for a class of p -Borel fixed ideals. This class includes the class of principal Cohen-Macaulay p -Borel fixed ideals. A minimal free resolution for Cohen-Macaulay p -Borel fixed ideals was first given by Aramova and Herzog [2]. Batzies proved in [5] that the resolution from [2] is even a cellular resolution.

In addition, we calculate the regularity of our class of p -Borel fixed ideals and give a formula for the multigraded Poincaré-Betti series $P_{S/\mathfrak{a}}^S(\underline{x}, t)$. The formula for the regularity is in terms of the minimal monomial generating system and hence can be applied to any p -Borel fixed ideal. We conjecture that this formula gives a lower bound for the regularity for general p -Borel fixed ideals. Our formula is a generalization of results obtained in [2] and [16].

3.1. Prerequisites on p -Borel Fixed Ideals. Before we come to the definition of p -Borel fixed ideals we give the context in which p -Borel fixed ideals arise. Just as Borel fixed ideals arise as the generic initial ideals of ideals in S when k has characteristic 0 the p -Borel fixed ideals arise as generic initial ideals when k has characteristic p . Even though therefore, p -Borel fixed ideals also constitute a very important class and have attracted quite some attention, they remain quite mysterious and many questions are still open. In particular, a minimal free resolution for general p -Borel fixed ideals is not known.

Let $p \in \mathbb{N}$ be a prime. Define the partial order $\leq_p$ on $\mathbb{N}$ by setting $a \leq_p b$ if and only if the p -adic expansions $a = \sum_{i=0}^s a_i p^i$, $0 \leq a_i \leq p-1$, and $b = \sum_{i=0}^s b_i p^i$, $0 \leq b_i \leq p-1$, fulfill $a_i \leq b_i$ for all $i = 1, \dots, s$.

Definition 6.13. A monomial ideal $\mathfrak{a} \trianglelefteq S$ is called p -Borel fixed if for all monomials $m \in \mathfrak{a}$ and all variables $x_i \in S$ we have

$$(6.3) \quad x_i^l \mid m, x_i^{l+1} \nmid m, j < i, \text{ and } \kappa \leq_p l \Rightarrow S_{ji}^\kappa(m) := \left(\frac{x_j}{x_i}\right)^\kappa m \in \mathfrak{a}.$$

It is known that this property only needs to be checked for the minimal monomial generating system of $\mathfrak{a}$ (see [2]).

Remark 6.14. As for Borel fixed ideals, there exists for each p -Borel fixed ideal a unique set $\mathfrak{B} = \{m_1, \dots, m_r\}$ such that $\mathfrak{B}$ generates the ideal $\mathfrak{a}$ minimally, in the sense that any monomial $m \in \mathfrak{a}$ has a decomposition $m = u v$ such that v is constructed from a monomial in $\mathfrak{B}$ by iterative applications of rule (6.3) to a monomial $m_r \in \mathfrak{B}$ (i.e. there exist numbers $i_1, \dots, i_l, j_1, \dots, j_l$, and $\kappa_1, \dots, \kappa_l$ with $j_t < i_t, t = 1, \dots, l$ such that $v = S_{j_1 i_1}^{\kappa_1}(\dots(S_{j_l i_l}^{\kappa_l}(m_r))\dots)$) and such that each proper subset $\mathfrak{B}'$ of $\mathfrak{B}$ generates a proper subideal $\mathfrak{a}'$ of $\mathfrak{a}$. Again, we call $\mathfrak{B}$ the Borel-generating system of $\mathfrak{a}$.

Definition 6.15 ([2]). A p -Borel fixed ideal $\mathfrak{a}$ is called principal p -Borel fixed if $\mathfrak{B}$ consists of only one monomial, i.e. $\mathfrak{B} = \{m\}$.

Lemma 6.16. If $\mathfrak{a} \trianglelefteq S$ is a principal p -Borel fixed ideal, then there exists a family of principal Borel fixed ideals $I_t, t = 1, \dots, T$, and numbers $0 \leq r_1 < r_2 < \dots < r_T$ such that

$$\mathfrak{a} = \prod_{j=1}^T I_j^{[p^{r_j}]}$$

where $I^{[q]}$ is the Frobenius power of an ideal, that is, if I is generated by $m_1, \dots, m_r$, then $I^{[q]}$ is generated by $m_1^q, \dots, m_r^q$.

PROOF. Assume $\mathfrak{B} = \{x_1^{a_1} \dots x_n^{a_n}\}$. Let $a_i = \sum_{j=0}^{l_i} a_{ij} p^j$ be the p -adic expansion of a_i . We define ideals I_j for $j = 1, \dots, n$:

$$I_j := \prod_{i=0}^n \langle x_1, \dots, x_i \rangle^{a_{ij}},$$

where we set $\langle x_1, \dots, x_i \rangle^{a_{ij}} := \langle 1 \rangle$ if $a_{ij} = 0$. Then it is straightforward to check (see [2]) that

$$\mathfrak{a} = \prod_{j=1}^n I_j^{[p^j]}.$$

Deleting the factors $I_j^{[p^j]} = \langle 1 \rangle^{[p^j]} = \langle 1 \rangle = S$ and relabeling the ideals I_j gives the numbers T and $0 \leq r_1 < \dots < r_T$. By Lemma 6.5 the ideals I_j are principal Borel fixed. $\square$

In general, the ideal $\mathfrak{a} = \prod_{j=1}^T I_j^{[p^{r_j}]}$ with $0 \leq r_1 < \dots < r_T$ and I_j Borel fixed for $j = 1, \dots, T$ is clearly p -Borel fixed. But it should be mentioned that not any p -Borel fixed ideal has such a decomposition.

From now on we consider the following class of p -Borel fixed ideals:

$$(6.4) \quad \mathfrak{a} = \prod_{j=1}^T I_j^{[p^{r_j}]} \text{ with } \begin{cases} 0 \leq r_1 < \dots < r_T \\ \deg(I_j) < p^{r_{j+1}-r_j}, j = 1, \dots, T-1 \\ I_j \text{ Borel fixed, } j = 1, \dots, T \end{cases}.$$

For an ideal $\mathfrak{a} \leq S$ with minimal monomial generating system $\text{MinGen}(\mathfrak{a})$ we define the total degree of $\mathfrak{a}$ by

$$\deg(\mathfrak{a}) := \max \{ \deg(m) \mid m \in \text{MinGen}(\mathfrak{a}) \}.$$

We further define for all $i = 1, \dots, n$

$$\deg_i(\mathfrak{a}) := \max \{ l \mid x_i^l \text{ divides a generator of } \mathfrak{a} \}.$$

Note that if $\mathfrak{a}$ is of type (6.4), we have for all $i = 1, \dots, T$

$$\sum_{j=1}^i \deg(I_j) p^{r_j} \leq \sum_{j=1}^i (p^{r_{j+1}-r_j} - 1) p^{r_j} = p^{r_{i+1}} - p^{r_1} < p^{r_{i+1}}.$$

Aramova and Herzog find in [2] explicit minimal free resolutions of principal p -Borel fixed ideals which are Cohen-Macaulay, which by [2] means that $\mathfrak{a}$ is generated by the power of a single variable $\mathfrak{B} = \{x_i^d\}$. Batzies proved in [5] that the resolution given in [2] is even cellular.

We develop a minimal cellular resolution for p -Borel fixed ideals $\mathfrak{a}$ of type (6.4) such that the factors I_j are principal Borel fixed for all $j = 1, \dots, T$. Clearly, this includes the case considered in [2] and [5]. In addition, we calculate regularity and the Poincaré-Betti series $P_{S/\mathfrak{a}}^S(\underline{x}, t)$ for these ideals.

We proceed by defining an acyclic matching on the Taylor resolution of $\mathfrak{a}$ such that the Morse complex is minimal. Since the Taylor resolution is cellular, it follows by [6] that the Morse complex is cellular, too. In order to do so, we have to introduce the following notation:

Notation:

- (1) $I_{<t} := \prod_{i=0}^{t-1} I_i$, $I_{>t} := \prod_{i=t+1}^T I_i$,
 $I_{<t}^{[p]} := \prod_{i=0}^{t-1} I_i^{[p^i]}$, $I_{>t}^{[p]} := \prod_{i=t+1}^T I_i^{[p^i]}.$
- (2) $M(u) := \max \{ i \mid x_i \text{ divides } u \}.$
- (3) For a monomial $u \in S$ we define $m_i(u) := \min \{ l > i \mid x_l \text{ divides } u \}.$
- (4) For all $j = 1, \dots, T$ let

$$\text{supp}(I_j) := \{ i \mid x_i \text{ divides } u \text{ for some } u \in \text{MinGen}(I_j) \}.$$

- (5) For a set $W := \{j_1 < \dots < j_\kappa\} \subset \{1, \dots, n\}$ we define the monomials $x_{t,\max}(W), x_{t,\min}(W) \in k[x_{j_1}, \dots, x_{j_\kappa}] =: k[W]$ such that $x_{t,\max}(W)$ and $x_{t,\min}(W)$ divide a generator of $I_{<t}^{[p]}$ and for all monomials $m \in k[W]$ dividing a generator of $I_{<t}^{[p]}$ we have either
 - $\deg(m) < \deg(x_{t,\max}(W))$ (resp. $\deg(m) < \deg(x_{t,\min}(W))$) or
 - $\deg(m) = \deg(x_{t,\max}(W))$ (resp. $\deg(m) = \deg(x_{t,\min}(W))$) and $m \preceq x_{t,\max}(W)$ (resp. $m \succeq x_{t,\min}(W)$),
where $\prec$ is the reverse lexicographic order.
- (6) For $j \in \{1, \dots, n\}$ we define the p -Borel fixed ideal

$$I_{<t}^{[p]}(j) := \langle u \in k[x_1, \dots, x_{j-1}] \text{ with } u x_{t,\max}(\{j, \dots, n\}) \in I_{<t}^{[p]} \rangle.$$

(7) We introduce the p -reverse lexicographic order:

$$m = \prod_{i=1}^T u_i^{p^{r_i}} \prec_t n = \prod_{i=1}^T v_i^{p^{r_i}}$$

if and only if there exists an index j such that $u_i = v_i$ for all $i > j$ and $u_j \prec v_j$.

Remark 6.17. If $\mathfrak{a}$ is Cohen-Macaulay, which by [2] is equivalent to $\mathfrak{a} = \prod_{i=1}^T (\langle x_1, \dots, x_d \rangle^{a_i})^{[p^{r_i}]}$, then we have always

$$\begin{aligned} x_{t,\max}(\{j_1 < \dots < j_\kappa\}) &= x_{j_1}^b, \\ x_{t,\min}(\{j_1 < \dots < j_\kappa\}) &= x_{j_l}^b \end{aligned}$$

where $b = \sum_{i=1}^{t-1} a_i p^{r_i}$ and $l = \max \{j \in \{j_1, \dots, j_\kappa\} \mid j \leq d\}$.

If $\mathfrak{a}$ is of type (6.4) such that the ideals I_j are principal Borel fixed we have

$$x_{t,\max}(\{j_1 < \dots < j_\kappa\}) = x_{j_1}^b \text{ with } b = \sum_{\substack{i=1 \\ j_1 \in \text{supp}(I_i)}}^{t-1} \deg_i(I_i) p^{r_i}.$$

From now on we assume that $\mathfrak{a}$ is of type (6.4) such that in addition the ideals I_j are principal Borel fixed. This implies that each generator in I_j has the same degree. We have the following decomposition:

Lemma 6.18. *Let $v \in I_{<_t}^{[p]}$ be a monomial generator and $W := \{j_1 < \dots < j_\kappa\} \subset \{1, \dots, n\}$. Then there exists a monomial v' dividing v such that*

- (1) $M(v') < j_1$ and
- (2) $v' x_{t,\max}(W)$ and $v' x_{t,\min}(W)$ are generators of $I_{<_t}^{[p]}$.

For a monomial $v \in I_{<_t}^{[p]}$ we let $\tilde{v}$ be the maximal monomial with respect to the order $\prec_t$ satisfying the properties (1) and (2).

PROOF. By definition the monomial v decomposes uniquely into a product $v = v_1 \cdots v_{t-1}$ such that $v_j \in I_j^{[p^{r_j}]}$. For all ideals I_j such that $\text{supp}(I_j) \cap W = \emptyset$, the rule (6.3) implies $M(v_j) < j_1$. In case $\text{supp}(I_j) \cap W \neq \emptyset$, we construct new monomials v'_i such that $M(v'_i) < j_1$.

Assume $\text{supp}(I_j) \cap W \neq \emptyset$. By definition I_j has the form

$$I_j = \langle x_1, \dots, x_{l_1} \rangle^{a_1} \cdots \langle x_1, \dots, x_{l_\kappa} \rangle^{a_\kappa}.$$

Therefore, v_j admits a decomposition $v_j = v_{j_1} \cdots v_{j_\kappa}$ with $v_{j_s} \in \langle x_1, \dots, x_{l_s} \rangle^{a_s}$. If $l_s < j_1$, we set $v'_{j_s} := v_{j_s}$, else we set $v'_{j_s} := 1$. We then define $v'_j := v'_{j_1} \cdots v'_{j_s}$ and $v' := v'_1 \cdots v'_{t-1}$. By construction we have $v' \mid v$ and $M(v') < j_1$. Finally, the definition of $x_{t,\max}$ and $x_{t,\min}$ implies that $v' x_{t,\min}$ and $v' x_{t,\max}$ are generators of the ideal $I_{<_t}^{[p]}$. $\square$

3.2. Construction of the Acyclic Matching.

We construct the matching on the Taylor resolution of $\mathfrak{a}$.

The construction of the matching is technical and proceeds in six steps. In order to allow the reader to keep track of the critical cells we provide a description of the remaining critical cells after each construction step.

Let $B := \left\{ v_0 u_0^{p^{r_t}} u, \dots, v_\kappa u_\kappa^{p^{r_t}} u \right\}$ be a basis element of the Taylor resolution such that $v_i \in I_{<t}^{[p]}$, $u_i \in I_t$ for $i = 1, \dots, \kappa$, and $u \in I_{>t}^{[p]}$. We assume that the elements $v_i u_i^{p^{r_t}} u$ are in increasing order with respect to the order $\prec_t$.

Now we define a monomial $w \in \mathfrak{a}$ and set

$$B \cup \{w\} \rightarrow B \setminus \{w\} \in \mathcal{M}.$$

Step 1: Let $0 < R \leq \kappa$ be the smallest index such that there exists an index $j < M(u_R)$ with either $\alpha_j > \beta_j + 1$ or $\beta_j > \alpha_j + 1$, where $u_0 = x^\alpha$ and $u_R = x^\beta$. In order to fulfill the matching property, we take the smallest index j with the desired properties.

- (1) If $\alpha_j > \beta_j + 1$, we set $w := v_R \left(\frac{x_j u_R}{x_{M(u_R)}} \right)^{p^{r_t}} u$. Then w divides $\text{lcm}(B)$ since the degree of x_j in w is the degree of x_j in v_R plus $p^{r_j} + \beta_j p^{r_j}$. Since $\deg(v_R) < p^{r_j}$, we get that the degree of x_j in w is smaller than $p^{r_j} + p^{r_j} + \beta_j p^{r_j} \leq \alpha_j p^{r_j}$.
- (2) If $\beta_j > \alpha_j + 1$, we set $w := v_0 \left(\frac{x_j u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u$ and with the same argumentation we get $w \mid \text{lcm}(B)$.

The critical cells in homological degree $\kappa + 1 \geq 1$ are given by

$$\mathcal{B}_{\kappa+1} := \left\{ v_0 u_0^{p^{r_t}} u, \dots, v_\kappa u_\kappa^{p^{r_t}} u \right\}$$

such that either $u_i = u_0$ or $u_i = u_0 \frac{x_{i_1} \cdots x_{i_r}}{x_{j_1} \cdots x_{j_r}}$ for some $i_1 < i_2 < \dots < i_r$ and $j_1, \dots, j_r$. Note that the indices $j_1, \dots, j_r$ are not necessarily pairwise different, consider for example $p = 11$, $u_0 = x_1^2 x_2^2 x_3^2 x_4^2 x_5$ and $u_1 = x_1^3 x_2^3 x_3^3$. But the increasing order implies $i_r < j_r \leq M(u_0)$.

Step 2: Let $B := \left\{ v_0 u_0^{p^{r_t}} u, \dots, v_\kappa u_\kappa^{p^{r_t}} u \right\}$ be a critical cell. We define $\alpha := \deg \left(\text{lcm} \{u_0, \dots, u_\kappa\} \right)$. Let i_0 be the smallest index such that the exponent of x_{i_0} in u_0 is smaller than α_{i_0} . Let j be the smallest index such that $u_j = u_0$ and $v_j \neq \tilde{v}_j x_{t, \max}(\{i_0\})$. We set

$$w := \tilde{v}_j x_{t, \max}(\{i_0\}) u_j^{p^{r_t}} u.$$

Since $M(\tilde{v}_j) < i_0$, $\tilde{v}_j \mid v_j$, and $x_{t, \max}(\{i_0\}) = x_{i_0}^b$ with $b < p^{r_t}$, we have $w \mid \text{lcm}(B)$.

The critical cells in homological degree $l_0 + \kappa \geq 1$ are then given by

$$\mathcal{B}_{l_0+\kappa} := \left\{ \begin{array}{l} v_{0l_0} x_{t, \max}(\{i_0\}) u_0^{p^t} u, \dots, v_{0l_0} x_{t, \max}(\{i_0\}) u_0^{p^t} u, \\ v_1 u_1^{p^{r_t}} u, \dots, v_\kappa u_\kappa^{p^{r_t}} u \end{array} \right\}$$

with $v_{0l_0}, \dots, v_{0l_0} \in I_{<t}^{[p]}(i_0)$, i_0 is the smallest index such that the exponent of x_{i_0} in u_0 is smaller than α_{i_0} , where $\alpha := \deg \left(\text{lcm} \{u_0, \dots, u_\kappa\} \right)$. For all $j = 1, \dots, \kappa$ we have $u_j = u_0 \frac{x_{\nu_1} \cdots x_{\nu_r}}{x_{\mu_1} \cdots x_{\mu_r}}$, for some $\nu_1 < \dots < \nu_r$ and $\mu_1, \dots, \mu_r \leq M(u_0)$ with $\nu_r < \mu_r$.

Step 3: Let $1 \leq R \leq \kappa$ be the smallest index such that $u_R = u_0 \frac{x_{i_1} \cdots x_{i_r}}{x_{j_1} \cdots x_{j_r}}$ for some $i_1 < i_2 < \dots < i_r, j_1, \dots, j_r$, and $r > 2$. Since $r > 2$, $i_r < j_r \leq M(u_0)$, and $i_0 \leq i_1$, there exists an index l such that $i_0 < i_l < j_r \leq M(u_0)$. Therefore, we can define $w := v_{01} x_{t, \max}(\{i_0\}) \left(\frac{x_{i_l} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u$. By construction we have $w \mid \text{lcm}(B)$.

The critical cells in homological degree $l_0 + \kappa \geq 1$ are now given by

$$\mathcal{B}_{l_0 + \kappa} := \left\{ \begin{array}{l} v_{01} x_{t, \max}(\{i_0\}) u_0^{p^{r_t}} u, \dots, v_{0l_0} x_{t, \max}(\{i_0\}) u_0^{p^{r_t}} u, \\ v_1 u_1^{p^{r_t}} u, \dots, v_\kappa u_\kappa^{p^{r_t}} u \end{array} \right\}$$

with $v_{01}, \dots, v_{0l_0} \in I_{< t}^{[p]}(i_0)$ such that for all $j = 1, \dots, \kappa$ we have $u_j = u_0 \frac{x_{\nu_j}}{x_{\mu_j}}$ for some $\nu_j < \mu_j \leq M(u_0)$ and we have $i_0 = \min \{ \nu_j \mid u_j = u_0 \frac{x_{\nu_j}}{x_{\mu_j}}, j = 1, \dots, \kappa \}$.

Step 4: Let B be a critical cell from Step 3 such that $u_j \in \left\{ \frac{x_{\nu_j} u_0}{x_{\mu_j}} \mid \nu_j < \mu_j \right\}$ and $i_0 := \min \{ \nu_j \mid 1 \leq j \leq \kappa \}$. We set $R := \min \{ \nu_j > i_0 \mid 1 \leq j \leq \kappa \}$, where we set $R := 0$ and $x_R := 1$ if all ν_j equal.

Case $R > 0$.

- (1) Let $u_j = \frac{x_{\nu_j} u_0}{x_{\mu_j}}$ such that $\nu_j = i_0$ and $v_j \neq \tilde{v}_j x_{t, \min}(\{R, \mu_j\})$. We define the monomial by $w := \tilde{v}_j x_{t, \min}(\{R, \mu_j\}) u_j^{p^{r_t}} u$. Since $\tilde{v}_j \mid v_j$, $M(\tilde{v}_j) < R$, $\deg(x_{t, \min}(\{R, \mu_j\})) < p^{r_t}$, and $u_j = u_0 \frac{x_{\nu_j} u_0}{x_{\mu_j}}$, we have $w \mid \text{lcm}(B)$.
- (2) If $u_j = \frac{x_{\nu_j} u_0}{x_{\mu_j}}$ with $\nu_j > i_0$ and $v_j \neq \tilde{v}_j x_{t, \min}(\{i_0, \mu_j\})$, we define $w := \tilde{v}_j x_{t, \min}(\{i_0, \mu_j\}) u_j^{p^{r_t}} u$, with $M(\tilde{v}_j) < i_0$. Since $\tilde{v}_j \mid v_j$, $M(\tilde{v}_j) < i_0$, $i_0 < \nu_j$, and $u_j = u_0 \frac{x_{\nu_j} u_0}{x_{\mu_j}}$, we have $w \mid \text{lcm}(B)$.
- (3) If neither (1) nor (2) is satisfied, we know that $v_j = \tilde{v}_j x_{t, \min}(\{i_0, \mu_j\})$. If there exists an index j such that $\mu_j < M(u_0)$, we define

$$w := \tilde{v}_j x_{t, \min}(\{i_0, M(u_0)\}) \left(\frac{x_{\nu_j} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u.$$

We always take the smallest index j with the desired properties.

Clearly, we have $w \mid \text{lcm}(B)$.

Case $R = 0$.

- (1) Let $u_j = \frac{x_{i_0} u_0}{x_{\mu_j}}$ such that $v_j \neq \tilde{v}_j x_{t, \min}(\{\mu_j\})$. We define the monomial w by $w := \tilde{v}_j x_{t, \min}(\{\mu_j\}) u_j^{p^{r_t}} u$. Since $\tilde{v}_j \mid v_j$, $M(\tilde{v}_j) < \mu_j$, $\deg(x_{t, \min}(\{\mu_j\})) < p^{r_t}$, and $u_j = u_0 \frac{x_{\nu_j} u_0}{x_{\mu_j}}$, we have $w \mid \text{lcm}(B)$.
- (2) If (1) is not satisfied, we have $v_j = \tilde{v}_j x_{t, \min}(\{\mu_j\})$ for all j . By definition of $m_{i_0}(u_0)$ we have $\mu_j \geq m_{i_0}(u_0)$ for all μ_j . If there exists an index j such that $\mu_j > m_{i_0}(u_0)$, we define

$$w := \tilde{v}_j x_{t, \min}(\{m_{i_0}(u_0)\}) \left(\frac{x_{i_0} u_0}{x_{m_{i_0}(u_0)}} \right)^{p^{r_t}} u$$

with $\tilde{v}_j \mid v_j$ and $M(\tilde{v}_j) < m_{i_0}(u_0)$.

Again, it is easy to see that $w \mid \text{lcm}(B)$.

The critical cells in homological degree $l_0 + l_1 \geq 1$ (resp. $l_0 + \dots + l_\kappa \geq 1$) are now given by

Type I:

$$\mathcal{B}_{l_0+l_1} = \left\{ \begin{array}{ll} v_{0\kappa} x_{t,\max}(\{i_1\}) u_0^{p^{r_t}} u, & \kappa = 1, \dots, l_0, \\ v_{1\kappa} x_{t,\min}(\{m_{i_1}(u_0)\}) \left(\frac{x_{i_1} u_0}{x_{m_{i_1}(u_0)}} \right)^{p^{r_t}} u, & \kappa = 1, \dots, l_1 \end{array} \right\}$$

with $v_{01}, \dots, v_{0l_0} \in I_{< t}^{[p]}(i_1)$ and $v_{11}, \dots, v_{1l_1} \in I_{< t}^{[p]}(m_{i_1}(u_0))$, $1 \leq i_1 < M(u_0)$.

Type II:

$$\mathcal{B}_{l_0+\dots+l_\kappa} = \left\{ \begin{array}{ll} v_{0j} x_{t,\max}(\{i_1\}) u_0^{p^{r_t}} u, & j = 1, \dots, l_0, \\ v_{1j} x_{t,\min}(\{i_2, M(u_0)\}) \left(\frac{x_{i_1} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u, & j = 1, \dots, l_1, \\ v_{2j} x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_2} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u, & j = 1, \dots, l_2, \\ \vdots & \\ v_{\kappa j} x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_\kappa} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u, & j = 1, \dots, l_\kappa \end{array} \right\}$$

with $1 \leq i_1 < \dots < i_\kappa < M(u_0)$, $v_{11}, \dots, v_{1l_1} \in I_{< t}^{[p]}(i_2)$, and $v_{j1}, \dots, v_{jl_j} \in I_{< t}^{[p]}(i_1)$ for all $j = 0, 2, \dots, \kappa$.

For a fixed index i we assume that the monomials $v_{i1}, \dots, v_{il_i}$ are in increasing order with respect to $\prec_t$.

Step 5: Consider a critical cell B of Step 4. For the monomial v_{11} we construct the monomial $\tilde{v}_{11} x_{t,\max}(\{i_1\})$ with $M(v'_{11}) < i_1$. If either $l_0 > 1$ or $l_0 = 1$ and $v_{01} \neq \tilde{v}_{11}$, we define $w := \tilde{v}_{11} x_{t,\max}(\{i_1\}) u_0^{p^{r_t}} u$. By construction we have $w \mid \text{lcm}(B)$.

The critical cells in homological degree $l_1 + 1 \geq 1$ (resp. $1 + l_1 + \dots + l_\kappa \geq 1$) are now given by

Type I:

$$\mathcal{B}_{1+l_1} = \left\{ \begin{array}{ll} v x_{t,\max}(\{i_1\}) u_0^{p^{r_t}} u, & \\ v_{1j} x_{t,\min}(\{m_{i_1}(u_0)\}) \left(\frac{x_{i_1} u_0}{x_{m_{i_1}(u_0)}} \right)^{p^{r_t}} u, & j = 1, \dots, l_1 \end{array} \right\},$$

$1 \leq i_1 < M(u_0)$ and $v_{11}, \dots, v_{1l_1} \in I_{< t}^{[p]}(m_{i_1}(u_0))$, and

Type II:

$$\mathcal{B}_{1+l_1+\dots+l_\kappa} = \left\{ \begin{array}{ll} v x_{t,\max}(\{i_1\}) u_0^{p^{r_t}} u, & \\ v_{1j} x_{t,\min}(\{i_2, M(u_0)\}) \left(\frac{x_{i_1} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u, & j = 1, \dots, l_1, \\ v_{2j} x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_2} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u, & j = 1, \dots, l_2, \\ \vdots & \\ v_{\kappa j} x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_\kappa} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u, & j = 1, \dots, l_\kappa \end{array} \right\}$$

with $1 \leq i_1 < \dots < i_\kappa < M(u_0)$, $v_{11}, \dots, v_{1l_1} \in I_{< t}^{[p]}(i_2)$, and $v_{j1}, \dots, v_{jl_j} \in I_{< t}^{[p]}(i_1)$ for all $j = 2, \dots, \kappa$.

In both cases $v = \min \{v' \mid v' x_{t,\min}(i_1) \text{ is a generator of } I_{<t}^{[p]}\}$.

For a fixed index i we assume that the monomials $v_{i_1}, \dots, v_{i_{l_i}}$ are in increasing order with respect to $<_t$.

Step 6: Consider a critical cell B of Step 5. Let $j_1 \in \{2, \dots, \kappa\}$ be the smallest index such that either $l_{j_1} > 1$ or $l_{j_1} = 1$ and $v_{j_1 1} \neq v$.

We define $w := v x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_{j_1}} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u$. Clearly, $w \mid \text{lcm}(B)$.

The critical cells in homological degree $l_1 + 1 \geq 1$ (resp. $\kappa + l_1 \geq 1$) are now given by

Type I:

$$\mathcal{B}_{1+l_1} = \left\{ \begin{array}{l} v x_{t,\max}(\{i_1\}) u_0^{p^{r_t}} u, \\ v_j x_{t,\min}(\{m_{i_1}(u_0)\}) \left(\frac{x_{i_1} u_0}{x_{m_{i_1}(u_0)}} \right)^{p^{r_t}} u, \quad j = 1, \dots, l_1 \end{array} \right\}$$

with $1 \leq i_1 < M(u_0)$ and $v_1, \dots, v_{l_1} \in I_{<t}^{[p]}(m_{i_1}(u_0))$ and

Type II:

$$\mathcal{B}_{\kappa+l_1} = \left\{ \begin{array}{l} v x_{t,\max}(\{i_1\}) u_0^{p^{r_t}} u, \\ v_j x_{t,\min}(\{i_2, M(u_0)\}) \left(\frac{x_{i_1} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u, \quad j = 1, \dots, l_1, \\ v x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_2} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u, \\ \vdots \\ v x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_\kappa} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u \end{array} \right\}$$

with $1 \leq i_1 < \dots < i_\kappa < M(u_0)$, $\kappa \geq 2$, and $v_1, \dots, v_{l_1} \in I_{<t}^{[p]}(i_2)$. In both cases $v = \min \{v' \mid v' x_{t,\min}(\{i_1\}) \text{ is a generator of } I_{<t}^{[p]}\}$.

The monomials $v_1, \dots, v_{l_1}$ are in increasing order with respect to $<_t$.

Proposition 6.19. *The matching $\mathcal{M}$ defined by the six steps above is a well defined acyclic matching.*

PROOF. The matching property as well as the invertibility are satisfied by construction. Now we prove acyclicity. We start with a set $\mathcal{B} = \{v_0 u_0^{p^{r_t}} u, \dots, v_\kappa u_\kappa^{p^{r_t}} u\}$ in homological degree $\kappa + 1$. We assume that the elements $v_i u_i^{p^{r_t}} u$ are in increasing order with respect to the order $<_t$.

Assume, that we have a directed cycle with elements in homological degree $\kappa + 1$ and $\kappa + 2$ starting with the element $\mathcal{B}$. In the first step we add a monomial w_1 constructed in Step 1, ..., Step 6. In the second step we have to remove a monomial $v_i u_i^{p^{r_t}} u$ from $\mathcal{B}$. In order to have a cycle we have to add this element again. Since $v_0 u_0^{p^{r_t}} u$ is the smallest monomial of $\mathcal{B}$, the construction of the monomials w in Step 1, ..., Step 6 implies that it can never be added in the cycle. Hence all sets occurring in the cycle have smallest element $v_0 u_0^{p^{r_t}} u$.

The construction of each monomial w in Step 1, ..., Step 6 depends either on the monomial u_0 or on the monomials u_0 and u_j for some j . Now we look at the cycle:

$$\mathcal{B} \rightarrow \mathcal{B} \cup \{w_1\} \rightarrow \mathcal{B} \cup \{w_1\} \setminus \{m_1\} \rightarrow \dots$$

Assume that w_1 is constructed with respect to u_0 and u_i for some i . If $m_1 \neq v_i u_i^{p^{r_i}} u$ then $\mathcal{B} \cup \{w_1\} \setminus \{m_1\}$ is matched with $\mathcal{B} \setminus \{m_1\}$. This contradicts the existence of the cycle. This reasoning shows that the existence of a cycle implies the existence of a sequence of added monomials $v_1, \dots, v_s$ such that v_i is constructed with respect to u_0 and v_{i+1} for all $i = 1, \dots, s-1$ and v_s is constructed with respect to u_0 and v_1 .

We prove that such a sequence cannot exist: Let $v_1, \dots, v_s$ be a sequence of added monomials such that v_i is constructed with respect to v_{i+1} . The following table lists the possible situations for these sequences, except Step 1, (1):

v_3	v_2	v_1
Step 5		
Step 6		
Step 4, $R = 0$, (ii)		
Step 4, $R = 0$, (i)	Step 4 $R = 0$, (ii)	
Step 4, $R > 0$, (i),(ii)	Step 4, $R = 0$, (ii)	
Step 4, $R > 0$, (iii)	Step 6	
Step 3	Step 4, $R > 0$, (iii)	Step 6
Step 2	Step 5	
Step 1 (2)	Step 4, $R > 0$, (iii)	Step 6
Step 1 (2)	Step 4, $R > 0$, (i),(ii)	Step 4, $R = 0$, (ii)

In none of these cases the sequence can consist of more than three monomials. If the sequence has exactly three monomials, the third monomial differs from the first monomial. The only step which can be applied more than once is Step 1 (1). But in this case the monomials are in increasing order with respect to $\prec_t$. Therefore, v_s can never be constructed with respect to v_1 . This makes such a cycle impossible and acyclicity follows. $\square$

3.3. Cellular resolutions and Poincaré-Betti Series for p -Borel Fixed Ideals.

In general, the Morse complex with respect to the acyclic matching from Proposition 6.19 is not minimal since the critical cells depend on the monomials $v_1, \dots, v_{l_1} \in I_{\prec_t}^{[p]}(i_1)$. But since the matching is constructed on the Taylor resolution, the Morse complex defines a cellular multigraded resolution of the ideal $\mathfrak{a}$. In the following case, which includes the principal Cohen-Macaulay p -Borel fixed ideals, the Morse complex is even minimal:

Theorem 6.20. *Let $\mathfrak{a} := \prod_{j=1}^T I_j^{[p^{r_j}]}$ be a p -Borel fixed ideal such that $I_j = \langle x_1, \dots, x_{l_j} \rangle^{a_j}$ with $a_j < p^{r_{j+1}-r_j}$ and $l_1 \geq l_2 \dots \geq l_T$.*

Then there is a minimal free resolution of $\mathfrak{a}$ that has the following basis in the i th ($i \geq 2$) homological degree:

$$S_i := \left\{ (u_t, \dots, u_T, \{j_1 < \dots < j_{i-1}\}) \mid t = 1, \dots, T, u_j \in I_j, j_{i-1} < M(u_t) \right\}.$$

Moreover the minimal free resolution is cellular.

Note that if $l_1 = \dots = l_T$, the ideal $\mathfrak{a}$ is principal Cohen-Macaulay p -Borel fixed and the sets S_i are exactly the bases of the minimal resolution given by Aramova and Herzog in [2].

PROOF. Consider a critical cell B from the matching of Proposition 6.19. In this case we have

$$\begin{aligned} x_{t,\min}(\{i_1, M(u_0)\}) &= x_{t,\min}(\{i_2, M(u_0)\}) = x_{M(u_0)}^b, \\ x_{t,\min}(\{m_{i_1}(u_0)\}) &= x_{m_{i_1}(u_0)}^b, \\ x_{t,\max}(\{i_1\}) &= x_{i_1}^b \end{aligned}$$

with $b = \sum_{j=1}^{t-1} a_j p^{r_j}$. Therefore, it follows $v = v_1 = \dots = v_{l_1} = 1$. This implies that the Morse complex has basis

Type I:

$$\mathcal{B}_2 = \left\{ x_{i_1}^b u_0^{p^{r_t}} u, x_{m_{i_1}(u_0)}^b \left(\frac{x_{i_1} u_0}{x_{m_{i_1}(u_0)}} \right)^{p^{r_t}} u \right\}$$

and

Type II:

$$\mathcal{B}_{\kappa+1} = \left\{ x_{i_1}^b u_0^{p^{r_t}} u, x_{M(u_0)}^b \left(\frac{x_{i_1} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u, \dots, x_{M(u_0)}^b \left(\frac{x_{i_\kappa} u_0}{x_{M(u_0)}} \right)^{p^{r_t}} u \right\}$$

with $i_1 < \dots < i_\kappa < M(u_0)$. A multidegree argument then implies that the Morse complex is minimal.

The basis elements depend only on $u_0 \in I_t^{[p]}$, $u \in I_{>t}^{[p]}$, and $i_1 < \dots < i_\kappa < M(u_0)$. Since u decomposes uniquely into a product $u = u_{t+1}^{p^{r_{t+1}}} \cdots u_T^{p^{r_T}}$ with $u_j \in I_j$, it follows that they are in bijection to the tuples $(u_t, \dots, u_T, \{i_1 < \dots < i_\kappa\})$ with the desired properties. This proves the assertion. $\square$

For general principle Borel fixed ideals I_j the Morse complex from 6.19 is not minimal, but we can still calculate the multigraded Poincaré-Betti series

$$P_{S/\mathfrak{a}}(\underline{x}, \tau) := \sum_{i, \alpha} \dim_\kappa (Tor_i^S(S/\mathfrak{a}, \kappa)_\alpha) \tau^i \underline{x}^\alpha.$$

For this we define

$$\overline{P}_{S/\mathfrak{a}}(\underline{x}, \tau) := \begin{cases} P_{S/\mathfrak{a}}(\underline{x}, \tau) - 1 & , \quad \mathfrak{a} \neq \langle \emptyset \rangle \\ 1 & , \quad \mathfrak{a} = \langle \emptyset \rangle. \end{cases}$$

We have the following formula to calculate the Poincaré-Betti series:

Theorem 6.21. *Let $\mathfrak{a} = \prod_{j=1}^T I_j^{[p^{r_j}]}$ be a p -Borel fixed ideal of type (6.4) such that the ideals I_j are principal Borel fixed. Then the multigraded Poincaré-Betti series is given by*

$$P_{S/\mathfrak{a}}(\underline{x}, \tau) = 1 + \sum_{u \in \text{MinGen}(\mathfrak{a})} u \tau + \sum_{t=1}^T \sum_{u \in G(I_{>t}^{[p]})} \sum_{u_t \in G(I_t)} u F(u_t)$$

with

$$\begin{aligned} F(u_t) &= \sum_{1 \leq i < M(u_t)} (x_i u_t)^{p^t} \tau \overline{P}_{S/I_{<t}^{[p]}(m_i(u_t))}(\underline{x}, \tau) \\ &+ \sum_{\substack{1 \leq i_1 < \dots < i_\kappa < M(u_t) \\ \kappa \geq 2}} (x_{i_1} \cdots x_{i_\kappa} u_t)^{p^t} \tau^\kappa \overline{P}_{S/I_{<t}^{[p]}(i_2)}(\underline{x}, \tau). \end{aligned}$$

In particular, the minimal free resolution is independent of the characteristic of k .

PROOF. For the proof, we apply the matching from Proposition 6.19 inductively. For that we have to argue that after a single application of the matching we essentially end up with a situation that allows to apply the matching again. In general, the Morse complex is not minimal after a single application, since the basis elements depend on the monomials $v_1, \dots, v_{l_1} \in I_{<t}^{[p]}(i_0)$. It is easy to see that $I_{<t}^{[p]}(i_0)$ is again a p -Borel fixed ideal. Thus, we can apply the same acyclic matching on the Taylor resolution of $I_{<t}^{[p]}(i_0)$ and consider in the original complex only monomials $v_1, \dots, v_{l_1}$ such that $\{v_1, \dots, v_{l_1}\}$ is a critical cell with respect to the acyclic matching on $I_{<t}^{[p]}(i_0)$. Since these critical cells depend again on some monomials in $I_{<t'}^{[p]}(i'_0)$, we can go on by induction. The induction is finite since $\text{supp}(I_{<t'}^{[p]}(i'_0)) \subsetneq \text{supp}(I_{<t}^{[p]}(i_0))$. Therefore, by applying the acyclic matching inductively, we reach a minimal cellular multigraded resolution of $\mathfrak{a}$. Because of the induction we are not able to give an explicit form of the resolution, but it proves that the resolution does not depend on the characteristic of k . Therefore, we can calculate the Poincaré-Betti series in the desired way. $\square$

The preceding proof admits the following corollary to Theorem 6.21. Indeed, its proof actually is a consequence of the proof of Theorem 6.21.

Corollary 6.22. *The ideal $\mathfrak{a} = \prod_{j=1}^T I_j^{[p^{r_j}]}$ of Theorem 6.21 admits a minimal multigraded cellular resolution.* $\square$

We explain the formula in two examples:

Example 6.23. Let $p = 2$ and $\mathfrak{a} := \langle x_1^2, x_2^2 \rangle \langle x_1^4, x_2^4, x_3^4 \rangle$. Using the computer algebra system CoCoA [13], we calculate the following minimal free resolution:

$$\longrightarrow S^3(-12) \longrightarrow S^4(-8) \oplus S^4(-10) \longrightarrow S^6(-6).$$

Our formula gives the following Poincaré-Betti series:

$$\begin{aligned} P_{S/\mathfrak{a}}(\underline{x}, \tau) &= 1 + \sum_{u \in G(I)} u \tau + x_1^2 x_2^2 x_1^4 \tau^2 + x_1^2 x_2^2 x_2^4 \tau^2 + x_1^2 x_2^2 x_3^4 \tau^2 + x_1^4 x_2^4 \tau^2 \\ &\quad + x_1^4 x_2^4 x_3^4 \tau^3 + x_1^4 x_3^4 \tau \bar{P}_{S/\langle x_1^2, x_2^2 \rangle}(\underline{x}, \tau) + x_2^4 x_3^4 \tau \bar{P}_{S/\langle x_1^2, x_2^2 \rangle}(\underline{x}, \tau) \end{aligned}$$

Thus, we obtain the following Poincaré-Betti series:

$$\begin{aligned} P_{S/\mathfrak{a}}(\underline{x}, \tau) &= 1 + \left(\underline{x}^{(6,0,0)} + \underline{x}^{(4,2,0)} + \underline{x}^{(2,4,0)} + \underline{x}^{(2,0,4)} + \underline{x}^{(0,6,0)} + \underline{x}^{(0,2,4)} \right) \tau \\ &\quad + \left(\underline{x}^{(6,2,0)} + \underline{x}^{(2,6,0)} + \underline{x}^{(2,2,4)} + \underline{x}^{(4,4,0)} \right) \tau^2 \\ &\quad + \left(\underline{x}^{(6,0,4)} + \underline{x}^{(4,2,4)} + \underline{x}^{(2,4,4)} + \underline{x}^{(0,6,4)} \right) \tau^2 \\ &\quad + \left(\underline{x}^{(6,2,4)} + \underline{x}^{(2,6,4)} + \underline{x}^{(4,4,4)} \right) \tau^3. \end{aligned}$$

Let $p = 3$ and $\mathfrak{a} := \langle x_1^3, x_2^3 \rangle (\langle x_1^2, x_1 x_2, x_1 x_3, x_2^2, x_2 x_3, x_3^2 \rangle)^9$. With CoCoA [13] we calculate the following minimal free resolution:

$$\longrightarrow S^3(-36) \oplus S^5(-33) \longrightarrow S^{10}(-30) \oplus S^3(-27) \oplus S^6(-24) \longrightarrow S^{12}(-21).$$

Our algorithm gives the following basis elements of the minimal free resolution:

$$t = 1, u_t = x_2 : \\ i = 1, \quad m_i(u) = 2 : \quad \{x_2^3 w, x_1^3 w\}, \quad w \in \{x_1^2, x_1 x_2, x_1 x_3, x_2^2, x_2 x_3, x_3^3\},$$

$$t = 2, u_t = x_1 x_2 : \\ i = 1, \quad m_i(u) = 2 : \quad \{x_1^3 x_1^9 x_2^9, x_2^3 x_1^{18}\},$$

$$t = 2, u_t = x_2^2 : \\ i = 1, \quad m_i(u) = 2 : \quad \{x_1^3 x_2^{18}, x_2^3 x_1^9 x_2^9\},$$

$$t = 2, u_t = x_1 x_3 : \\ \begin{array}{lll} i = 1, & m_i(u) = 3 : & \{x_1^3 x_1^9 x_3^9, w x_1^{18}\}, & w \in \{x_1^3, x_2^3\}, \\ i = 2, & m_i(u) = 3 : & \{x_2^3 x_1^9 x_3^9, w x_1^9 x_2^9\}, & w \in \{x_1^3, x_2^3\}, \\ i = 1, & m_i(u) = 3 : & \{x_1^3 x_1^9 x_3^9, w_1 x_1^{18}, w_2 x_1^{18}\}, & w_{1,2} \in \{x_1^3, x_2^3\}, \\ i = 2, & m_i(u) = 3 : & \{x_2^3 x_1^9 x_3^9, w_1 x_1^9 x_2^9, w_2 x_1^9 x_2^9\}, & w_{1,2} \in \{x_1^3, x_2^3\}, \\ i_1 = 1, i_2 = 2, & m_i(u) = 3 : & \{x_2^3 x_1^9 x_3^9, x_2^3 x_1^{18}, x_1^3 x_1^9 x_2^9\}, \end{array}$$

$$t = 2, u_t = x_3^2 : \\ \begin{array}{lll} i = 1, & m_i(u) = 3 : & \{x_1^3 x_3^{18}, w x_1^9 x_3^9\}, & w \in \{x_1^3, x_2^3\}, \\ i = 2, & m_i(u) = 3 : & \{x_2^3 x_3^{18}, w x_2^9 x_3^9\}, & w \in \{x_1^3, x_2^3\}, \\ i = 1, & m_i(u) = 3 : & \{x_1^3 x_3^{18}, w_1 x_1^9 x_3^9, w_2 x_1^9 x_3^9\}, & w_{1,2} \in \{x_1^3, x_2^3\}, \\ i = 2, & m_i(u) = 3 : & \{x_2^3 x_3^{18}, w_1 x_2^9 x_3^9, w_2 x_2^9 x_3^9\}, & w_{1,2} \in \{x_1^3, x_2^3\}, \\ i_1 = 1, i_2 = 2, & m_i(u) = 3 : & \{x_1^3 x_3^{18}, x_2^3 x_1^9 x_3^9, x_1^3 x_2^9 x_3^9\}, \end{array}$$

$$t = 2, u_t = x_2 x_3 : \\ \begin{array}{lll} i = 1, & m_i(u) = 2 : & \{x_1^3 x_2^9 x_3^9, x_2^3 x_1^9 x_3^9\}, \\ i = 2, & m_i(u) = 3 : & \{x_2^3 x_2^9 x_3^9, w x_2^{18}\}, & w \in \{x_1^3, x_2^3\}, \\ i = 2, & m_i(u) = 3 : & \{x_2^3 x_2^9 x_3^9, w_1 x_2^{18}, w_2 x_2^{18}\}, & w_{1,2} \in \{x_1^3, x_2^3\}, \\ i_1 = 1, i_2 = 2, & m_i(u) = 3 : & \{x_1^3 x_2^9 x_3^9, x_2^3 x_1^9 x_2^9, x_1^3 x_2^{18}\}. \end{array}$$

Counting basis elements gives the desired Poincaré-Betti series.

3.4. Regularity of p -Borel Fixed Ideals.

Finally, we study the regularity of p -Borel fixed ideals. Recall that the regularity of an ideal $\mathfrak{a}$ is the maximal number j such that

$$\text{Tor}_i^S(S/\mathfrak{a}, k)_{i+j} \neq 0 \text{ for some } i \geq 0.$$

Let $u = \prod_{i=1}^n x^{\mu_i}$ be a monomial in S and $\mu_i := \sum_j \mu_{ij} p^j$ the p -adic expansion of μ_i . For a real number $x \in \mathbb{R}$ denote by $\lfloor x \rfloor$ the largest integer $n \leq x$. For $1 \leq \kappa \leq n$ and $j \geq 0$ we define

$$d_{\kappa j}(\mu) = \sum_{i=1}^{\kappa} \lfloor \frac{\mu_i}{p^j} \rfloor.$$

For each κ with $\mu_\kappa \neq 0$ we set $s_\kappa := \lfloor \log_p \mu_\kappa \rfloor$ and set

$$D_\kappa := d_{\kappa, s_\kappa}(\mu) p^{s_\kappa} + (\kappa - 1)(p^{s_\kappa} - 1).$$

Finally, we define the following functions:

Definition 6.24. (1) For a monomial $u \in S$ which is not divisible by x_1 we set

$$pa(u) := \max_{\kappa: \mu_\kappa \neq 0} \{D_\kappa\}.$$

(2) For a monomial $u = x_1^{\mu_1} \tilde{u} \in S$ such that $\tilde{u}$ is not divisible by x_1 we set

$$pa(u) := \mu_1 + pa(\tilde{u}).$$

(3) For a p -Borel fixed ideal of type (6.4) we define the following function:

$$\begin{aligned} pa_t(\mathfrak{a}) &:= \sum_{j=t+1}^T \deg(I_j) p^{r_j} + \max_{u \in G(I_t)} \{\deg(u) + (M(u) - 1)(p^{r_t} - 1)\}, \\ pa(\mathfrak{a}) &:= \max_{t=1, \dots, T} \{pa_t(\mathfrak{a})\}. \end{aligned}$$

The function $pa(\mathfrak{a})$ was first introduced by Pardue [35], who conjectured that if $\mathfrak{a}$ is p -Borel fixed, Borel-generated by $\mathfrak{B} := \{x^\mu\}$ such that x_1 does not divide x^μ , then

$$\text{reg}(\mathfrak{a}) = pa(x^\mu).$$

Herzog and Popescu proved this conjecture in [23]. In addition, they proved an upper bound for p -Borel ideals $\mathfrak{a}$ Borel-generated by $\mathfrak{B} := \{u_1, \dots, u_r\}$.

Theorem 6.25 (see [23]). *If $\mathfrak{a}$ is p -Borel fixed, Borel-generated by $\mathfrak{B} := \{u_1, \dots, u_r\}$, then*

$$\text{reg}(\mathfrak{a}) \leq \max \{pa(u_1), \dots, pa(u_r)\}$$

and equality holds if $\mathfrak{a}$ is principal p -Borel fixed.

In [16], Ene, Pfister and Popescu calculate the regularity of p -Borel fixed ideals in the following case:

Theorem 6.26 (see [16]). *Let $\mathfrak{a}$ be a p -Borel fixed ideal of type (6.4) such that I_j contains $x_m^{p^{r_{j+1}} - r_j - 1}$ for all $j = 1, \dots, T$ where $m := \max(\text{supp}(I_{j+1}))$. Then*

$$\text{reg}(\mathfrak{a}) = pa(\mathfrak{a}).$$

We now give a generalization of Theorem 6.26.

Theorem 6.27. *Let $\mathfrak{a} := \prod_{j=1}^T I_j^{[p^{r_j}]}$ be a p -Borel fixed ideal such that $I_j = \langle x_1, \dots, x_{l_j} \rangle^{a_j}$ with $a_j < p^{r_{j+1} - r_j}$ for all $j = 1, \dots, T$, and $l_1 \geq l_2 \dots \geq l_T$. Then*

$$\text{reg}(\mathfrak{a}) = pa(\mathfrak{a}).$$

PROOF. By Theorem 6.20 a basis of $\text{Tor}_i^S(S/\mathfrak{a}, k)$ is given by

$$S_{i-1} := \left\{ (u_t, \dots, u_T, \{j_1 < \dots < j_{i-1}\}) \mid t = 1, \dots, T, u_j \in I_j, j_{i-1} < M(u_t) \right\},$$

where the multidegree is given by

$$\deg((u_t, \dots, u_T, \{j_1 < \dots < j_{i-1}\})) = \sum_{j=t+1}^T \deg(I_j) p^{r_j} + (\deg(u_t) p^{r_t} + (i-1) p^{r_t}).$$

Therefore, the basis element $(u_t, \dots, u_T, \{j_1 < \dots < j_{i-1}\})$ gives a contribution to $\text{Tor}_{i-1}^S(S/\mathfrak{a}, k)_{i-1+j}$ with

$$j = \sum_{l=t+1}^T \deg(I_l) p^{r_l} + (\deg(u_t) p^{r_t} + (i-1)(p^{r_t} - 1)).$$

Clearly, this becomes maximal if $i = M(u_t) - 1$ and if $\deg(u_t)$ is maximal. Taking the maximum over $t = 1, \dots, T$ proves the assertion. $\square$

If $\mathfrak{a}$ is a p -Borel fixed ideal of type (6.4) such that I_j is principal Borel fixed, we have the following regularity:

Theorem 6.28. *Let $\mathfrak{a} = \prod_{j=1}^T I_j^{[p^{r_j}]}$ be a principal p -Borel fixed ideal of type (6.4) such that the factors I_j are principal Borel fixed. Then*

$$\text{reg}(\mathfrak{a}) = \max_{t=1, \dots, T} \left\{ pa_t(\mathfrak{a}) + \text{reg}(I_{<t}^{[p]}(2)) \right\} \geq pa(\mathfrak{a}).$$

Note that it is actually possible to deduce Theorem 6.27 and the equality in Theorem 6.25 from Theorem 6.28: In the situation of Theorem 6.27 we have $I_{<t}^{[p]}(2) = \emptyset$ and therefore $\text{reg}(I_{<t}^{[p]}(2)) = 0$.

If $\mathfrak{B} = \{x^\mu\}$ is the Borel-generating system of $\mathfrak{a}$, Lemma 6.16 implies that the regularity satisfies $\text{reg}(I_{<t}^{[p]}(2)) = \mu_1$. Therefore, this gives a new proof of the equality of Theorem 6.25:

Corollary 6.29. *If $\mathfrak{a}$ is Borel-generated by $\mathfrak{B} = \{x^\mu\}$, we have:*

$$\text{reg}(\mathfrak{a}) = \max_{t=1, \dots, T} \left\{ pa_t(\mathfrak{a}) + \text{reg}(I_{<t}^{[p]}(2)) \right\} = pa(\mathfrak{a}) + \mu_1.$$

In particular, if $\mathfrak{a}$ is Borel-generated by $\mathfrak{B} = \{x^\mu\}$ and x_1 does not divide x^μ , we have $\text{reg}(\mathfrak{a}) = pa(\mathfrak{a}) = pa(x^\mu)$, which proves Pardue's conjecture. $\square$

PROOF OF THEOREM 6.28. The set of critical cells with respect to the matching constructed in Proposition 6.19 give a basis for $\text{Tor}_i^S(S/\mathfrak{a}, k)_{i+l}$. It follows that we get the following possible l 's:

$$\sum_{j=t+1}^T \deg(I_j)p^{r_j} + \max_{u_t \in G(I_t)} \left(\deg(u_t)p^{r_t} + \max_{\substack{1 \leq i_1 < \dots < i_\kappa < M(u_t) \\ \kappa \geq 2}} \{ \kappa(p^{r_t} - 1) + \text{reg}(I_{<t}^{[p]}(i_2)) \} \right),$$

$$\sum_{j=t+1}^T \deg(I_j)p^{r_j} + \max_{u_t \in G(I_t)} \left(\deg(u_t)p^{r_t} + \max_{1 \leq i_1 < M(u_t)} \{ (p^{r_t} - 1) + \text{reg}(I_{<t}^{[p]}(m_{i_1}(u_t))) \} \right).$$

Since $\text{reg}(I_{<t}^{[p]}(i_2)) < p^{r_t}$, maximality is achieved for $\kappa = M(u) - 1$. Then $i_2 = 2$ and therefore $I_{<t}^{[p]}(i_2)$ has only one generator, namely x_1^b for some b . It follows that the regularity of $I_{<t}^{[p]}$ is b . Thus, we have

$$\sum_{j=t+1}^T \deg(I_j)p^{r_j} + \max_{u_t \in G(I_t)} \left(\deg(u_t)p^{r_t} + (M(u_t) - 1)(p^{r_t} - 1) + \text{reg}(I_{<t}^{[p]}(2)) \right).$$

Taking the maximum over $t = 1, \dots, T$ proves the assertion. $\square$

Example 6.30. Let $p = 2$ and $\mathfrak{a} := \langle x_1 \rangle^{p^0} \langle x_1, x_2, x_3 \rangle^{p^1}$. Then it follows: $pa(\mathfrak{a}) = 4 = pa_2(\mathfrak{a})$. The program Macaulay 2 [20] calculates $\text{reg}(\mathfrak{a}) = 5$. Our formula gives:

$$\begin{aligned} \text{reg}(\mathfrak{a}) &= \max_{t=1,2} \left\{ pa_t(\mathfrak{a}) + \text{reg}(I_{<t}^{[p]}(2)) \right\} \\ &= 4 + \text{reg}(\langle x_1 \rangle^{[p^0]}) \\ &= 4 + 1 = 5. \end{aligned}$$

Note that the acyclic matching $\mathcal{M}$ from Proposition 6.19 can be applied in a slightly more general setting. Let $\mathfrak{a}$ be any p -Borel fixed ideal of type (6.4). Then Lemma 6.18 does not hold anymore, but one can prove that for each monomial v there exists a generator v_1 such that v_1 divides $\tilde{v} x_{t,\min}(W)$ (resp. $\tilde{v} x_{t,\max}(W)$). One can construct the matching in the same way, but instead of $B \cup \{w\} \rightarrow B \setminus \{w\}$ one defines $B \cup \{\tilde{w}\} \rightarrow B \setminus \{\tilde{w}\}$ where $\tilde{w}$ is the smallest generator of $\mathfrak{a}$ dividing w . The same arguments as used in the proof of Proposition 6.19 imply that the matching is still acyclic. In this case, we cannot anymore describe the critical cells. But we get the same multidegrees since the smallest generator (with respect to a chosen monomial order) dividing $\frac{x_j u_0}{x_{M(u_0)}}$ must have the same exponent for x_j , otherwise it would divide u_0 , which is impossible since u_0 is a generator. Furthermore, the induction principle is still valid, hence we finally reach a minimal free resolution, which is cellular since it comes from the Taylor resolution by a single matching. Therefore, Theorem 6.21, Corollary 6.22 and Theorem 6.28 are still true in this situation:

Theorem 6.31. *If $\mathfrak{a}$ is any p -Borel fixed ideal of type (6.4), then:*

- (1) *$\mathfrak{a}$ admits a minimal cellular multigraded resolution.*
- (2) *The multigraded Poincaré-Betti series $P_{S/\mathfrak{a}}(\underline{x}, \tau)$ is given by the formula of Theorem 6.21.*
- (3) *The regularity is given by*

$$\text{reg}(\mathfrak{a}) = \max_{t=1, \dots, T} \left\{ pa_t(\mathfrak{a}) + \text{reg}(I_{<t}^{[p]}(2)) \right\}.$$

In particular, the minimal free resolution does not depend on the characteristic of k . $\square$

In general, we would like to conjecture the following:

Conjecture 6.32. *If $\mathfrak{a} = \prod_{j=1}^T I_j^{[p^{r_j}]}$ is p -Borel fixed such that there exists an index j with $\deg(I_j) \geq p^{r_{j+1}-r_j}$, then*

$$\max_{t=1, \dots, T} \left\{ pa_t(\mathfrak{a}) + \text{reg}(I_{<t}^{[p]}(2)) \right\} \leq \text{reg}(\mathfrak{a}).$$

APPENDIX A

The Bar and the Hochschild Complex

In this chapter we show how to obtain the normalized Bar resolution (resp. the normalized acyclic Hochschild complex) via algebraic discrete Morse theory from the Bar resolution (resp. the acyclic Hochschild complex). Since our applications are restricted to quotients of the free associative algebra, we again consider the Bar resolution (resp. the acyclic Hochschild complex) only for k -algebras A . Note, the proofs still hold for R -algebras A , where R is a commutative Ring and A is projective as an R -module.

For the general definition of the Bar resolution (resp. the acyclic Hochschild complex). See [47, Chapter 8.1] and [10, Chapter 2.11] for more background information.

The motivation for adding this Appendix is twofold. First, it serves as a sample application of algebraic discrete Morse theory in a non-trivial but still transparent situation. Second, it also shows that applications of algebraic discrete Morse theory yield proofs of homotopy equivalences of chain complexes, whose chain homotopies are essentially the natural ones if the matching is chosen by a simple and uniform combinatorial rule.

Let A be a k -algebra and let W be a basis of A as a k -vector space such that $1 \in W$ and M an A -module. The Bar resolution

$$\mathcal{B}_\bullet : \cdots \xrightarrow{\partial_{i+1}} \mathcal{B}_i \xrightarrow{\partial_i} \mathcal{B}_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_1} \mathcal{B}_1 \xrightarrow{\partial_1} \mathcal{B}_0 = k$$

of M with respect to W is defined by:

- (1) $\mathcal{B}_i$ is the free $(A \otimes_k M)$ -module with basis elements

$$[w_1 | \dots | w_i] \text{ for } w_1, \dots, w_i \in W.$$

- (2) The differential ∂_i is given by

$$\begin{aligned} \partial([w_1 | \dots | w_i]) &= (w_1 \otimes 1) [w_2 | \dots | w_i] \\ &+ \sum_{j=1}^{i-1} (-1)^j \left((a_0 \otimes 1) [w_1 | \dots | w_{j-1} | 1 | w_{j+2} | \dots | w_i] \right. \\ &\quad \left. + \sum_l (a_l \otimes 1) [w_1 | \dots | w_{j-1} | w'_l | w_{j+2} | \dots | w_i] \right) \\ &+ (-1)^i (1 \otimes w_i) [w_1 | \dots | w_{i-1}], \end{aligned}$$

$$\text{if } w_j w_{j+1} = a_0 + \sum_l a_l w'_l \text{ with } a_0, a_l \in k \text{ and } w'_l \in W.$$

Proposition A.1 (Normalized Bar Resolution). *Let A be a k -algebra and let W be a basis of A as a k -vector space such that $1 \in W$, and let M be an A -module. Then there is an acyclic matching $\mathcal{M}$ on the Bar resolution $\mathcal{B}_A$ with respect to W such that the corresponding Morse complex $\mathcal{B}^{\mathcal{M}}$ is given by:*

- (1) $\mathcal{B}_i^{\mathcal{M}}$ is the free $(A \otimes M)$ -module with basis elements

$$[w_1 | \dots | w_i] \text{ for } w_1, \dots, w_i \in W \setminus \{1\}.$$

(2) The Mores differential $\partial_i^{\mathcal{M}}$ is given by

$$\begin{aligned} \partial^{\mathcal{M}}([w_1 | \dots | w_i]) &= (w_1 \otimes 1) [w_2 | \dots | w_i] \\ &+ \sum_{j=1}^{i-1} (-1)^j \sum_l (a_l \otimes 1) [w_1 | \dots | w_{j-1} | w'_l | w_{j+2} | \dots | w_i] \\ &+ (-1)^i (1 \otimes w_i) [w_1 | \dots | w_{i-1}] \end{aligned}$$

if $w_j w_{j+1} = a_0 + \sum_l a_l w'_l$ with $a_0, a_l \in k$ and $w'_l \in W \setminus \{1\}$.

In particular, $\mathcal{B}^{\mathcal{M}} = \mathbf{NB}$ is the normalized Bar resolution.

PROOF. We define the matching $\mathcal{M}$ by

$$[w_1 | \dots | w_l | w_{l+1} | \dots | w_i] \rightarrow [w_1 | \dots | w_l w_{l+1} | \dots | w_i] \in \mathcal{M}$$

if $w_l := \min(j \mid w_j = 1)$, $w_{l'} := \max(j \mid w_r = 1 \text{ for all } l \leq r \leq j)$, and $l' - l$ is odd. The invertibility is given since in both cases the coefficient in the differential is ± 1 :

$$\partial([w_1 | \dots | w_l | w_{l+1} | \dots | w_i]) = \pm 1 [w_1 | \dots | w_l w_{l+1} | \dots | w_i].$$

It is easy to see that the other conditions of an acyclic matching are satisfied as well. The critical cells are exactly the desired basis elements and an element $[w_1 | \dots | w_i]$ for which $w_j = 1$ for some j is never mapped to an element $[w_1 | \dots | w_i]$, with $w_j \neq 1$ for all j . This implies the formula for the Morse differential. $\square$

Let A be a k -algebra and let W be a basis of A as a k -vector space such that $1 \in W$. The acyclic Hochschild complex

$$\mathrm{HC}_A : \dots \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 = k$$

with respect to W is then given by:

(1) C_i is the free $(A \otimes_k A)$ -module with basis elements

$$[w_1 | \dots | w_i] \text{ for } w_1, \dots, w_i \in W,$$

(2) the differential ∂_i is given by

$$\begin{aligned} \partial_i([w_1 | \dots | w_i]) &= (w_1 \otimes 1) [w_2 | \dots | w_i] + (-1)^i (1 \otimes w_i) [w_1 | \dots | w_{i-1}] \\ &+ \sum_{j=1}^{i-1} (-1)^j \left(\begin{array}{l} a_0 [w_1 | \dots | w_{j-1} | 1 | w_{j+2} | \dots | w_i] \\ + \sum_l a_l [w_1 | \dots | w_{j-1} | w'_l | w_{j+2} | \dots | w_i] \end{array} \right) \end{aligned}$$

if $w_j w_{j+1} = a_0 + \sum_l a_l w'_l$, with $a_0, a_l \in k$ and $w'_l \in W \setminus \{1\}$.

Proposition A.2 (Normalized Acyclic Hochschild Complex). *Let A be a k -algebra and let W be a basis of A as a k -vector space such that $1 \in W$, and let M be an A -module. Then there is an acyclic matching $\mathcal{M}$ on the acyclic Hochschild complex HC_A of A such that the corresponding Morse complex $\mathrm{HC}^{\mathcal{M}}$ is given by:*

(1) $C_i^{\mathcal{M}}$ is the free $(A \otimes_k A)$ -module with basis elements

$$[w_1 | \dots | w_i] \text{ for } w_1, \dots, w_i \in W \setminus \{1\}.$$

(2) The Morse differential $\partial_i^{\mathcal{M}}$ is given by

$$\begin{aligned} \partial_i([w_1 | \dots | w_i]) &= (w_1 \otimes 1) [w_2 | \dots | w_i] + (-1)^i (1 \otimes w_i) [w_1 | \dots | w_{i-1}] \\ &+ \sum_{j=1}^{i-1} (-1)^j \left(\sum_l a_l [w_1 | \dots | w_{j-1} | w'_l | w_{j+2} | \dots | w_i] \right) \end{aligned}$$

if $w_j w_{j+1} = a_0 + \sum_l a_l w'_l$ with $a_0, a_l \in k$ and $w'_l \in W \setminus \{1\}$.

In particular, $\mathrm{HC}^{\mathcal{M}}$ is the normalized acyclic Hochschild complex.

PROOF. The proof is essentially identical to the proof of Proposition A.1. $\square$

APPENDIX B

Proofs for Algebraic Discrete Morse Theory

In this chapter we give our prove of the algebraic discrete Morse theory (Theorem 2.2). We assume the notation from Chapter 2. The proof will be split in several lemmas and is based on the construction of an explicit homotopy of chain complexes. In addition, this homotopy allows the description of the differential of the Morse complex in combinatorial term analogous to the differential in (topological) discrete Morse theory [17] and [18].

We write $\Gamma_{\downarrow}(c, c')$ (resp. $\Gamma_{\uparrow}(c, c')$) for the sum of the weights of all those paths from c to c' for which the first step $c \rightarrow c_1$ satisfies $c \in X_i^{\mathcal{M}}$ and $c_1 \in X_{i-1}^{\mathcal{M}}$ (resp. $c' \in X_{i+1}^{\mathcal{M}}$). In most cases it will be clear from the context, e.g. if c is critical, whether the first step increases or decreases dimension. Still for the sake of readability we will always equip Γ with the respective arrow.

From now on we assume always that $\mathcal{M}$ satisfies the three conditions.

We first prove that the Morse differential satisfies $\partial_i^{\mathcal{M}} \circ \partial_{i+1}^{\mathcal{M}} = 0$.

Lemma B.1. *Let $\mathcal{M} \subset E$ be an acyclic matching on $G(C_{\bullet}) = (V, E)$. Then*

(P1) *$\partial^{\mathcal{M}}$ is a differential (i.e. $\partial^{\mathcal{M}} \circ \partial^{\mathcal{M}} = 0$).*

(P2) *For $(\alpha, \beta, [\alpha : \beta]) \in \mathcal{M}$ with $\alpha \in X_{i+1}$, $\beta \in X_i$ we have for all $c \in X_{i-1}^{\mathcal{M}}$:*

$$\Gamma_{\downarrow}(\beta, c) = \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(\beta, c') \Gamma_{\downarrow}(c', c).$$

PROOF. The proof is by induction over the cardinality of $\mathcal{M}$. In order to prove the induction, we assume that both properties are satisfied for smaller matchings.

Let $\mathcal{M} = \{(\alpha, \beta, [\alpha : \beta])\}$ be a matching of cardinality 1.

Property (P2):

$$\begin{aligned} 0 = \partial^2(\alpha) &= \sum_{c' \in X_i^{\mathcal{M}}} [\alpha : c'] \partial(c') + [\alpha : \beta] \partial(\beta) \\ &= \sum_{c \in X_{i-1}} \left(\sum_{c' \in X_i^{\mathcal{M}}} [\alpha : c'] [c' : c] \right) c \\ &\quad + \sum_{c \in X_{i-1}} [\alpha : \beta] [\beta : c] c \\ &= -[\alpha : \beta] \sum_{c \in X_{i-1}} \left(\sum_{c' \in X_i^{\mathcal{M}}} \left(-\frac{1}{[\alpha : \beta]} \right) [\alpha : c'] [c' : c] \right) c \\ &\quad + \sum_{c \in X_{i-1}} [\alpha : \beta] [\beta : c] c \end{aligned}$$

$$\begin{aligned}
&= -[\alpha : \beta] \sum_{c \in X_{i-1}} \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(\beta, c') \Gamma_{\downarrow}(c', c) c \\
&\quad + \sum_{c \in X_{i-1}} [\alpha : \beta] [\beta : c] c \\
&= [\alpha : \beta] \sum_{c \in X_{i-1}} \left([\beta : c] - \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(\beta, c') \Gamma_{\downarrow}(c', c) \right) c \\
&= [\alpha : \beta] \sum_{c \in X_{i-1}} \left(\Gamma_{\downarrow}(\beta, c) - \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(\beta, c') \Gamma_{\downarrow}(c', c) \right) c.
\end{aligned}$$

Since $[\alpha : \beta] \in Z(R) \cap R^*$ is not a zero-divisor and the critical cells are linearly independent, we get the desired result:

$$\Gamma_{\downarrow}(\beta, c) - \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(\beta, c') \Gamma_{\downarrow}(c', c) = 0.$$

Property (P1): Let $c \in X_{i+1}^{\mathcal{M}}$ be a critical cell. We have to distinguish three cases. Note that the validity of property (P2) has been established above.

Case 1: $(\partial^{\mathcal{M}})^2(c) = \partial^2(c)$. Since ∂ is a differential, we are done.

Case 2: There exist elements $\beta \in X_i$ and $c \neq \alpha \in X_{i+1}$ with

$$[c : \beta] \neq 0 \text{ and } \{(\alpha, \beta, [\alpha : \beta])\} = \mathcal{M}.$$

Then we have:

$$\begin{aligned}
(\partial^{\mathcal{M}})^2(c) &= \sum_{\beta \neq c' \leq c} [c : c'] \partial^{\mathcal{M}}(c') + [c : \beta] \left(-\frac{1}{[\alpha : \beta]} \right) \sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} [\alpha : c'] \partial^{\mathcal{M}}(c') \\
&= \sum_{\beta \neq c' \leq c} \sum_{c'' \leq c'} [c : c'] [c' : c''] c'' \\
&\quad + [c : \beta] \left(-\frac{1}{[\alpha : \beta]} \right) \sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} \sum_{c'' \leq c'} [\alpha : c'] [c' : c''] c'' \\
&= \sum_{c'' \in X_{i-1}^{\mathcal{M}}} \left(\sum_{\beta \neq c' \leq c} [c : c'] [c' : c''] \right. \\
&\quad \left. + [c : \beta] \left(-\frac{1}{[\alpha : \beta]} \right) \sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} [\alpha : c'] [c' : c''] \right) c'' \\
&= \sum_{c'' \in X_{i-1}^{\mathcal{M}}} \left(\sum_{\beta \neq c' \leq c} [c : c'] [c' : c''] + [c : \beta] \sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} \Gamma_{\uparrow}(\beta, c') \Gamma_{\downarrow}(c', c'') \right) c'' \\
&\stackrel{(P2)}{=} \sum_{c'' \in X_{i-1}^{\mathcal{M}}} \left(\sum_{\beta \neq c' \leq c} [c : c'] [c' : c''] + [c : \beta] \Gamma_{\downarrow}(\beta, c'') \right) c''
\end{aligned}$$

$$= \sum_{c'' \in X_{i-1}} \left(\sum_{c' \leq c} [c : c'] [c' : c''] \right) c'' = \partial^2(c) = 0.$$

Case 3: There exist elements $\beta \in X_i$ and $\alpha \in X_{i-1}$ with

$$[c : \beta] \neq 0 \text{ and } \{(\beta, \alpha, [\beta : \alpha])\} = \mathcal{M}.$$

Since $\partial^2(c) = 0$, we have

$$\begin{aligned} 0 &= \sum_{c' \leq c} [c : c'] [c' : \alpha] \\ &= [c : \beta] [\beta : \alpha] + \sum_{\substack{c' \leq c \\ c' \neq \beta}} [c : c'] [c' : \alpha] \\ &= [\beta : \alpha] \left([c : \beta] + \sum_{\substack{c' \leq c \\ c' \neq \beta}} \frac{1}{[\beta : \alpha]} [c : c'] [c' : \alpha] \right). \end{aligned}$$

Since $[\beta : \alpha] \in Z(R) \cap R^*$ is not a zero-divisor, it follows

$$(B.1) \quad [c : \beta] = \sum_{\substack{c' \leq c \\ c' \neq \beta}} \left(-\frac{1}{[\beta : \alpha]} \right) [c : c'] [c' : \alpha].$$

This observation allows the deduction of the desired result:

$$\begin{aligned} (\partial^{\mathcal{M}})^2(c) &= \sum_{\substack{c' \leq c \\ c' \neq \beta}} [c : c'] \partial^{\mathcal{M}}(c') \\ &= \sum_{\substack{c' \leq c \\ c' \neq \beta}} \sum_{\substack{c'' \leq c' \\ c'' \neq \alpha}} [c : c'] [c' : c''] c'' \\ &\quad + \sum_{\substack{c'' \leq \beta \\ c'' \neq \alpha}} \underbrace{\left(\sum_{\substack{c' \leq c \\ c' \neq \beta}} [c : c'] [c' : \alpha] \left(-\frac{1}{[\beta : \alpha]} \right) \right)}_{=[c : \beta] \text{ by (B.1)}} [\beta : c''] c'' \\ &= \sum_{\substack{c' \leq c \\ c' \neq \beta}} \sum_{\substack{c'' \leq c' \\ c'' \neq \alpha}} [c : c'] [c' : c''] c'' \\ &\quad + \sum_{\substack{c'' \leq \beta \\ c'' \neq \alpha}} [c : \beta] [\beta : c''] c'' \\ &= \sum_{c' \leq c} \sum_{\substack{c'' \leq c' \\ c'' \neq \alpha}} [c : c'] [c' : c''] c'' = 0 \text{ since } \partial^2 = 0. \end{aligned}$$

We now assume properties (P1) and (P2) for matchings of cardinality $\leq n$. Let $\mathcal{M}$ be an acyclic matching of cardinality $n + 1$, and $\mathcal{M}' := \mathcal{M} \setminus \{(\alpha, \beta, [\alpha : \beta])\}$ with $\alpha \in X_{i+1}^{\mathcal{M}'}$ and $\beta \in X_i^{\mathcal{M}'}$. Then α, β are critical with respect to $\mathcal{M}'$, and by induction $\mathcal{M}'$ satisfies (P1) and (P2).

Property (P2):

$$\begin{aligned}
0 &= (\partial^{\mathcal{M}'})^2(\alpha) = \sum_{c' \in X_i^{\mathcal{M}'}} \sum_{c \leq \alpha} [\alpha : c] \Gamma_{\uparrow}(c, c') \partial^{\mathcal{M}'}(c') \\
&= [\alpha : \beta] \partial^{\mathcal{M}'}(\beta) + \sum_{\substack{c' \in X_i^{\mathcal{M}'} \\ c' \neq \beta}} \sum_{c \leq \alpha} [\alpha : c] \Gamma_{\uparrow}(c, c') \partial^{\mathcal{M}'}(c') \\
&= [\alpha : \beta] \partial^{\mathcal{M}'}(\beta) + \sum_{\substack{c' \in X_i^{\mathcal{M}'} \\ c' \neq \beta}} \Gamma_{\downarrow}(\alpha, c') \partial^{\mathcal{M}'}(c') \\
&= [\alpha : \beta] \left(\sum_{c' \leq \beta} [\beta : c'] \sum_{c \in X_{i-1}^{\mathcal{M}'}} \Gamma_{\uparrow}(c', c) c \right) \\
&\quad + \sum_{\substack{c' \in X_i^{\mathcal{M}'} \\ c' \neq \beta}} \Gamma_{\downarrow}(\alpha, c') \sum_{c \in X_{i-1}^{\mathcal{M}'}} \Gamma_{\downarrow}(c', c) c \\
&= [\alpha : \beta] \left(\sum_{c \in X_{i-1}^{\mathcal{M}'}} \Gamma_{\downarrow}(\beta, c) c \right) \\
&\quad - [\alpha : \beta] \sum_{c \in X_{i-1}^{\mathcal{M}'}} \left(-\frac{1}{[\alpha : \beta]} \right) \sum_{\substack{c' \in X_i^{\mathcal{M}'} \\ c' \neq \beta}} \Gamma_{\downarrow}(\alpha, c') \Gamma_{\downarrow}(c', c) c \\
&= [\alpha : \beta] \sum_{c \in X_{i-1}^{\mathcal{M}'}} (\Gamma_{\downarrow}(\beta, c) - \sum_{c' \in X_i^{\mathcal{M}'}} \Gamma_{\uparrow}(\beta, c') \Gamma_{\downarrow}(c', c)) c.
\end{aligned}$$

Since the critical cells are linearly independent and $[\alpha : \beta]$ is a unit, we get the desired result:

$$\Gamma_{\downarrow}(\beta, c) - \sum_{c' \in X_{\mathcal{M}}^{(i)}} \Gamma_{\uparrow}(\beta, c') \Gamma_{\downarrow}(c', c) = 0$$

Property (P1): Let $c \in X_{i+1}^{\mathcal{M}}$ be a critical cell. In order to prove the first statement, we have, as in the case of cardinality 1, to distinguish three cases:

Case 1: $(\partial^{\mathcal{M}})^2(c) = (\partial^{\mathcal{M}'})^2(c)$. Since by induction $(\partial^{\mathcal{M}'})^2 = 0$, we are done.

Case 2: There exist elements $c \neq \alpha \in X_{i+1}^{\mathcal{M}'}$ and $\beta \in X_i^{\mathcal{M}'}$ with

$$[c : \beta] \neq 0 \text{ and } (\alpha, \beta, [\alpha : \beta]) \in \mathcal{M}.$$

Then we have:

$$\partial^{\mathcal{M}}(c) = \sum_{c' \in X_i^{\mathcal{M}}} [c : \beta] \Gamma_{\uparrow}(\beta, c') c' + \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\downarrow}(c, c') c',$$

where the last sum is over all paths which do not pass through β . It follows

$$\begin{aligned}
(\partial^{\mathcal{M}})^2(c) &= \sum_{c' \in X_i^{\mathcal{M}}} [c : \beta] \Gamma_{\uparrow}(\beta, c') \partial^{\mathcal{M}}(c') + \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\downarrow}(c, c') \partial^{\mathcal{M}}(c') \\
&= \sum_{c'' \in X_{i-1}^{\mathcal{M}}} [c : \beta] \left(\sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(\beta, c') \Gamma_{\downarrow}(c', c'') \right) c'' + \sum_{c'' \in X_{i-1}^{\mathcal{M}}} \Gamma_{\downarrow}(c, c'') c''
\end{aligned}$$

$$\begin{aligned}
&= \sum_{c'' \in X_{i-1}^{\mathcal{M}}} [c : \beta] \Gamma_{\downarrow}(\beta, c'') c'' + \sum_{c'' \in X_{i-1}^{\mathcal{M}}} \Gamma_{\downarrow}(c, c'') c'' \\
&= \sum_{c'' \in X_{i-1}^{\mathcal{M}'}} \Gamma_{\downarrow}(c, c'') c'' = 0 \text{ since by induction } \partial^{\mathcal{M}'} \circ \partial^{\mathcal{M}'} = 0.
\end{aligned}$$

Case 3: There exist elements $\beta \in X_i^{\mathcal{M}'}$ and $\alpha \in X_{i-1}^{\mathcal{M}'}$ with

$$[c : \beta] \neq 0 \text{ and } (\beta, \alpha, [\beta : \alpha]) \in \mathcal{M}.$$

Then we have

$$\begin{aligned}
(\partial^{\mathcal{M}})^2(c) &= \sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} \Gamma_{\downarrow}(c, c') \partial^{\mathcal{M}}(c') \\
&= \sum_{c'' \neq \alpha} \left(\sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} \Gamma_{\downarrow}(c, c') [c' : \alpha] \left(-\frac{1}{[\beta : \alpha]} \right) \right) \Gamma_{\downarrow}(\beta, c'') c'' \\
&\quad + \sum_{c'' \neq \alpha} \left(\sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} \Gamma_{\downarrow}(c, c') \Gamma_{\downarrow}(c', c'') \right) c'', \text{ where } \Gamma_{\downarrow}(c', c'') \text{ does not pass} \\
&\quad \text{through } \alpha \\
&\stackrel{(*)}{=} \sum_{c'' \neq \alpha} \Gamma_{\downarrow}(c, \beta) \Gamma_{\downarrow}(\beta, c'') c'' + \sum_{c'' \neq \alpha} \left(\sum_{\substack{c' \in X_{i-1}^{\mathcal{M}'} \\ c' \neq \beta}} \Gamma_{\downarrow}(c, c') \Gamma_{\downarrow}(c', c'') \right) c'' \\
&= \sum_{c'' \neq \alpha} \left(\sum_{c' \in X_i^{\mathcal{M}'}} \Gamma_{\downarrow}(c, c') \Gamma_{\downarrow}(c', c'') \right) c'' \\
&= 0 \text{ since } (\partial^{\mathcal{M}'})^2 = 0.
\end{aligned}$$

In (*) we use the fact $\Gamma_{\downarrow}(c, \beta) = \sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} \Gamma_{\downarrow}(c, c') \left(-\frac{1}{[\beta : \alpha]} \right) [c' : \alpha]$, which holds

with the same argument as in (B.1). $\square$

In the following, we show that the Morse complex is homotopy equivalent to the original complex. Thereby, it will be possible to minimize a complex of free R -modules by means of algebraic discrete Morse theory.

Let $(C(X, R), \partial)$ be a complex of free R -modules, $\mathcal{M} \subset E$ a matching on the associated graph $G(C(X, R)) = (V, E)$, and $(C(X^{\mathcal{M}}, R), \partial^{\mathcal{M}})$ the Morse complex. We consider the following maps:

$$\begin{aligned}
\text{(B.2)} \quad f : C(X, R) &\rightarrow C(X^{\mathcal{M}}, R) \\
c \in X_i &\mapsto f(c) := \sum_{c' \in X_i^{\mathcal{M}}} \Gamma(c, c') c',
\end{aligned}$$

$$\text{(B.3)} \quad g : C(X^{\mathcal{M}}, R) \rightarrow C(X, R)$$

$$\begin{aligned}
c \in X_i^{\mathcal{M}} &\mapsto g_i(c) := \sum_{c' \in X_i} \Gamma(c, c') c', \\
\chi : C(X, R) &\rightarrow C(X, R) \\
c \in X_i &\mapsto \chi_i(c) := \sum_{c' \in X_{i+1}} \Gamma(c, c') c'.
\end{aligned}
\tag{B.4}$$

Then:

Lemma B.2. *The maps f and g are homomorphisms of complexes of free R -modules. In particular,*

$$\begin{aligned}
\text{(C1)} \quad &\partial^{\mathcal{M}} \circ f = f \circ \partial, \\
\text{(C2)} \quad &\partial \circ g = g \circ \partial^{\mathcal{M}}.
\end{aligned}$$

Lemma B.3. *The maps g and f define a chain homotopy. In particular,*

$$\begin{aligned}
\text{(H1)} \quad &g_i \circ f_i - \text{id} = \partial \circ \chi_{i+1} + \chi_i \circ \partial, \text{ i.e. it is null-homotopic,} \\
\text{(H2)} \quad &f_i \circ g_i - \text{id} = 0; \text{ in particular, } f \circ g \text{ is null-homotopic.}
\end{aligned}$$

Corollary B.4 (Thm. 2.2). *$C(X^{\mathcal{M}}, R)$ is a complex of free R -modules and*

$$H_i(C(X, R), R) = H_i(C(X^{\mathcal{M}}, R), R) \text{ for all } i \geq 0.$$

PROOF. This is an immediate consequence of Lemma B.3. $\square$

PROOF OF LEMMA B.2: **Property (C1):** Let $c \in X_i$. Then:

$$(\partial^{\mathcal{M}} \circ f)(c) = \partial^{\mathcal{M}} \left(\sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(c, c') c' \right) = \sum_{c'' \in X_{i-1}^{\mathcal{M}}} \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(c, c') \Gamma_{\downarrow}(c', c'') c''$$

and

$$\begin{aligned}
(f \circ \partial)(c) &= f \left(\sum_{c' \leq c} [c : c'] c' \right) \\
&= \sum_{c'' \in X_{i-1}^{\mathcal{M}}} \sum_{c' \leq c} [c : c'] \Gamma_{\uparrow}(c', c'') c'' \\
&= \sum_{c'' \in X_{i-1}^{\mathcal{M}}} \Gamma_{\downarrow}(c, c'').
\end{aligned}$$

Using Lemma B.1 (P2) the assertion now follows.

Property (C2): Let $c \in X_i^{\mathcal{M}}$. Then:

$$\begin{aligned}
(\partial \circ g)(c) &= \sum_{c'' \leq c} [c : c''] c'' + \sum_{c' \in X_i} \Gamma_{\downarrow}(c, c') \sum_{c'' \leq c'} [c' : c''] c'' \\
&= \underbrace{\sum_{c'' \in X_{i-1}^{\mathcal{M}}} \Gamma_{\downarrow}(c, c'') c''}_{(A)} \\
&\quad + \underbrace{\sum_{c' \in X_i} \Gamma_{\downarrow}(c, c') \sum_{\substack{c'' \leq c' \\ (c'', \beta, [c'': \beta]) \in \mathcal{M}}} [c' : c''] c''}_{(B)}
\end{aligned}$$

$$+ \underbrace{\sum_{c' \in X_i} \Gamma_{\downarrow}(c, c') \sum_{\substack{c'' \leq c' \\ (\beta, c'', [\beta : c'']) \in \mathcal{M}}} [c' : c''] c''}_{(C)}.$$

We have $(C) = 0$: Fix $c'' \in X_{i-1}$ and $\beta \in X_i$ such that $(\beta, c'', [\beta : c'']) \in \mathcal{M}$. Then:

$$\begin{aligned} \sum_{c' \in X_i} \Gamma_{\downarrow}(c, c') [c' : c''] &= \sum_{c' \neq \beta} \Gamma_{\downarrow}(c, c') [c' : c''] c'' + \Gamma_{\downarrow}(c, \beta) [\beta : c''] c'' \\ &= \sum_{c' \neq \beta} \Gamma_{\downarrow}(c, c') [c' : c''] c'' \\ &\quad + \left(\Gamma_{\downarrow}(c, c'') \left(-\frac{1}{[\beta : c'']} \right) \right) [\beta : c''] c'' \\ &= \Gamma_{\downarrow}(c, c'') c'' - \Gamma_{\downarrow}(c, c'') c'' = 0. \end{aligned}$$

On the other hand:

$$\begin{aligned} (g \circ \partial^{\mathcal{M}})(c) &= g \left(\sum_{c' \in X_{i-1}^{\mathcal{M}}} \Gamma_{\downarrow}(c : c') c' \right) \\ &= \underbrace{\sum_{c' \in X_{i-1}^{\mathcal{M}}} \Gamma_{\downarrow}(c, c') c'}_{(A)} + \underbrace{\sum_{c' \in X_{i-1}^{\mathcal{M}}} \sum_{\substack{c'' \in X_{i-1} \\ (\beta, c'', [\beta : c'']) \in \mathcal{M}}} \Gamma_{\downarrow}(c, c') \Gamma_{\downarrow}(c', c'') c''}_{(D)}. \end{aligned}$$

We will verify $(B) = (D)$: Consider the matching $\mathcal{M}' \setminus \{(c'', \beta, [c'', \beta])\}$. Since c'' and β are critical cells in $\mathcal{M}'$, it follows by Lemma B.1 (P1) (i.e. $(\partial^{\mathcal{M}'})^2 = 0$) that

$$0 = \sum_{\substack{c' \in X_{i-1}^{\mathcal{M}'} \\ c' \neq c''}} \Gamma_{\downarrow}(c, c') \Gamma_{\downarrow}(c', \beta) + \Gamma_{\downarrow}(c, c'') [c'' : \beta].$$

Multiplying by $\left(-\frac{1}{[c'' : \beta]} \right)$ yields:

$$\sum_{c' \in X_{i-1}^{\mathcal{M}'}} \Gamma_{\downarrow}(c, c') \Gamma_{\downarrow}(c', c'') = \sum_{c' \in X_i} \Gamma_{\downarrow}(c, c') [c' : c''].$$

Thus, $(B) = (D)$. $\square$

PROOF OF LEMMA B.3: Property (H2): Let $c \in X_i^{\mathcal{M}}$. The map g sends c to a sum over all $c' \in X_i$ that can be reached from c . Since c is critical, c' can be reached from c if either $c = c'$ or there is a $c'' \in X_{i-1}$ such that $(c', c'', [c' : c'']) \in \mathcal{M}$. Moreover,

$$f(c) = 0 \text{ if there is a } c' \in X_{i-1} \text{ such that } (c, c', [c : c']) \in \mathcal{M}.$$

Since f and g are R -linear, it follows that $(f_i \circ g_i)(c) = f_i(c)$. From $f_X^{\mathcal{M}} = \text{id}$ we infer the assertion.

Property (H1): We distinguish three cases.

Case 1: Assume c is critical. Then

$$(g_i \circ f_i - \text{id})(c) = g_i(c) - c = \sum_{\substack{c' \in X_i \\ (c', \beta, [c' : \beta]) \in \mathcal{M}}} \Gamma_{\downarrow}(c, c') c'.$$

Moreover, $\chi_i(c) = 0$, in particular, $(\partial \circ \chi_i)(c) = 0$.

$$\begin{aligned}
\chi(\partial(c)) &= \chi \left(\sum_{c' \leq c} [c : c'] c' \right) \\
&= \sum_{c' \leq c} [c : c'] \sum_{c'' \in X_i} \Gamma_{\uparrow}(c', c'') c'' \\
&= \sum_{\substack{c'' \in X_i \\ (c'', \beta, [c' : \beta]) \in \mathcal{M}}} \Gamma_{\downarrow}(c, c'') c'' = (g_i \circ f_i - \text{id})(c).
\end{aligned}$$

Case 2: There is an $\alpha \in X_{i-1}$ such that $(c, \alpha, [c : \alpha]) \in \mathcal{M}$. Then $\chi(c) = 0$ and $(g_i \circ f_i - \text{id})(c) = -\text{id}(c) = -c$. Moreover,

$$\begin{aligned}
\chi(\partial(c)) &= \chi \left(\sum_{c' \leq c} [c : c'] c' \right) \\
&= \sum_{c' \leq c} [c : c'] \sum_{c'' \in X_i} \Gamma_{\uparrow}(c', c'') c'' \\
&= [c : \alpha] \left(-\frac{1}{[c : \alpha]} \right) c \\
&\quad + \sum_{\substack{c' \leq c \\ c' \neq \alpha}} [c : c'] \sum_{\substack{c'' \in X_i \\ c'' \neq \alpha}} \Gamma_{\uparrow}(c', c'') c'' + [c : \alpha] \sum_{c'' \in X_i} \Gamma_{\uparrow}(\alpha, c'') c''.
\end{aligned}$$

Since

$$\Gamma_{\uparrow}(\alpha, c'') = \left(-\frac{1}{[c : \alpha]} \right) \sum_{\substack{c' \leq c \\ c' \neq \alpha}} [c : c'] \Gamma_{\uparrow}(c', c''),$$

the assertion follows.

Case 3: There is an $\alpha \in X_{i+1}$ such that $(\alpha, c, [\alpha : c]) \in \mathcal{M}$. Then:

$$\begin{aligned}
(g_i \circ f_i - \text{id})(c) &= -c + \underbrace{\sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(c, c') c'}_{(A)} \\
&\quad + \underbrace{\sum_{\substack{c'' \in X_i \\ (c'', \beta, [c'' : \beta]) \in \mathcal{M}}} \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(c, c') \Gamma_{\downarrow}(c', c'') c''}_{(B)}.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\partial \chi(c) &= \partial \left(\sum_{c' \in X_{i+1}} \Gamma_{\uparrow}(c, c') c' \right) \\
&= \sum_{c' \in X_{i+1}} \Gamma_{\uparrow}(c, c') \sum_{c'' \leq c'} [c' : c''] c''
\end{aligned}$$

$$\begin{aligned}
&= \sum_{c'' \leq \alpha} \left(-\frac{1}{[\alpha : c]} \right) [\alpha : c''] c'' \\
&\quad + \sum_{c' \neq \alpha} \Gamma_{\uparrow}(c, c') \sum_{c'' \leq c'} [c' : c''] c'' \\
&= -c + (A) + \underbrace{\sum_{\substack{c'' \in X_i \\ (c'', \beta, [c'' : \beta]) \in \mathcal{M}}} \sum_{c' \neq \alpha} \Gamma_{\uparrow}(c, c') \sum_{c'' \leq c'} [c' : c''] c''}_{(C)} \\
&\quad + \underbrace{\sum_{\substack{c'' \in X_i \\ (\beta, c'', [\beta : c'']) \in \mathcal{M}}} \sum_{c' \neq \alpha} \Gamma_{\uparrow}(c, c') \sum_{c'' \leq c'} [c' : c''] c''}_{(D)}
\end{aligned}$$

and

$$\begin{aligned}
\chi \partial(c) &= \chi \left(\sum_{c' \leq c} [c : c'] c' \right) \\
&= \sum_{\substack{c'' \in X_i \\ (c'', \beta, [c'' : \beta]) \in \mathcal{M}}} \sum_{c' \leq c} [c : c'] \Gamma_{\uparrow}(c', c'') c'' \\
&= \underbrace{\sum_{\substack{c'' \in X_i \\ (c'', \beta, [c'' : \beta]) \in \mathcal{M}}} \Gamma_{\downarrow}(c, c'') c''}_{(E)}.
\end{aligned}$$

We show:

- (a) $(D) = 0$,
- (b) $(E) + (C) = (B)$.

Assertion (a); Fix $c'' \in X_i$ and $\beta \in X_{i+1}$ such that $(\beta, c'', [\beta : c'']) \in \mathcal{M}$. Then:

$$\begin{aligned}
&\sum_{c' \in X_{i+1}} \Gamma_{\uparrow}(c, c') [c' : c''] \\
&= \sum_{c' \neq \beta} \Gamma_{\uparrow}(c, c') [c' : c''] c'' + \Gamma_{\uparrow}(c, \beta) [\beta : c''] \\
&= \sum_{c' \neq \beta} \Gamma_{\uparrow}(c, c') [c' : c''] c'' \\
&\quad + \left(\Gamma_{\uparrow}(c, c'') \left(-\frac{1}{[\beta : c'']} \right) \right) [\beta : c''] c'' \\
&= \Gamma_{\uparrow}(c, c'') c'' - \Gamma_{\uparrow}(c, c'') c'' = 0.
\end{aligned}$$

Assertion (b); Let $c'' \in X_i$ and $\beta \in X_{i-1}$ such that $(c'', \beta, [c'' : \beta]) \in \mathcal{M}$. Consider the matching $\mathcal{M}' = \mathcal{M} \setminus \{(c'', \beta, [c'' : \beta])\}$. Then by Lemma B.1 (P2)

$$\sum_{c' \in X_i^{\mathcal{M}'}} \Gamma_{\uparrow}(c, c') \Gamma_{\downarrow}(c', \beta) = \Gamma_{\downarrow}(c, \beta).$$

Since c'' is critical with respect to $\mathcal{M}'$, it follows that

$$\sum_{\substack{c' \in X_i^{\mathcal{M}'} \\ c' \neq c''}} \Gamma_{\uparrow}(c, c') \Gamma_{\downarrow}(c', \beta) + \Gamma_{\uparrow}(c, c'')[c'' : \beta] = \Gamma_{\downarrow}(c, \beta).$$

Multiplying the equation with $\left(-\frac{1}{[c'' : \beta]}\right)$ yields

$$\sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(c, c') \Gamma_{\downarrow}(c', c'') = \sum_{c' \in X_{i+1}} \Gamma_{\uparrow}(c, c')[c' : c''] + \Gamma_{\downarrow}(c, c''),$$

where paths are taken with respect to the matching $\mathcal{M}$. Hence $(B) = (C) + (E)$. $\square$

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